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Generalized 2-microlocal Besov spaces

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Gutachter

1. Prof. Hans-Jürgen Schmeißer
2. Prof. Hans-Gerd Leopold
3. Prof. Miroslav Krbec

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Zusammenfassung

Das Konzept 2-mikrolokaler Analysis oder 2-mikrolokaler Funktionräume wurde von Jean-Michel Bony [Bo84] und Stéphane Jaffard entwickelt [Ja91]. Es ist ein gutes Instrument in der Nähe von Singularitäten lokale Regularität und oszillierendes Verhalten von Funktionen zu beschreiben.

Der Ansatz ist fourieranalytisch und nutzt Littlewood-Paley Analysis von Distributionen. Solche Räume zu untersuchen wurde zuerst von Jaak Peetre auf Seite 266 in [Pe75] vorgeschlagen. Die Theorie wurde in Spezialfällen von vielen Autoren ausgearbeitet und fand Anwendung in Fraktaler Analysis und der Signalverarbeitung. Zu erwähnen sind dabei die Arbeiten [Ja91], [JaMey96], [LVSeu04], [Mey97], [MeyXu97], [MoYa04] und [Xu96].

Die größten Erkenntnisse wurden durch Wavelet Methoden, die Wavelet Charakterisierung der 2 mikrolokalen Räume, ermöglicht. In dieser Arbeit beschreiben wir einen vereinheitlichten fourieranalytischen Ansatz, um die 2-mikrolokalen Räume zu 2-mikrolokalen Besovräumen zu verallgemeinern, und wir interessieren uns für lokale sowie Wavelet Charakterisierungen dieser Räume.

Dafür sei $\{\varphi_j\}_{j\in\mathbb{N}_0}$ eine glatte Zerlegung der Eins (siehe Abschnitt 2.2 für die präzise Definition) und sei $\{w_j\}_{j\in\mathbb{N}_0}$ eine Folge zulässiger Gewichtsfunktionen. Diese genügen

$$\begin{aligned} 0 < w_j(x) &\le \mathrm{C} w_j(y)(1+2^j|x-y|)^\alpha \\ 2^{-\alpha_1} w_j(x) &\le w_{j+1}(x) \le 2^{\alpha_2} w_j(x) \ , \end{aligned}$$

für $x, y \in \mathbb{R}^n$, $j \in \mathbb{N}_0$ und $\alpha, \alpha_1, \alpha_2 \ge 0$. Weiterhin seien $\mathcal{F}$ und $\mathcal{F}^{-1}$ die Fouriertransformation und ihre inverse im Raum der Distributionen $\mathcal{S}'(\mathbb{R}^n)$. Wir definieren die Räume $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ für $0 < p, q \le \infty$ und $s \in \mathbb{R}$ als Menge aller $f \in \mathcal{S}'(\mathbb{R}^n)$ für die gilt

$$\left\| f \middle| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| w_j \mathcal{F}^{-1}(\varphi_j \mathcal{F} f) \middle| L_p(\mathbb{R}^n) \right\|^q \right)^{1/q} < \infty \tag{0.1}$$

für $0 < q < \infty$ und

$$\left\| f \middle| B^{s,mloc}_{p\infty}(\mathbb{R}^n, \boldsymbol{w}) \right\| = \sup_{j\in\mathbb{N}_0} 2^{js} \left\| w_j \mathcal{F}^{-1}(\varphi_j \mathcal{F} f) \middle| L_p(\mathbb{R}^n) \right\| < \infty \ , \tag{0.2}$$

für $q = \infty$. Als Spezialfall setzen wir $w_j(x) = (1 + 2^j|x - x_0|)^{s'}$ für $s' \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$ und $j \in \mathbb{N}_0$. Falls $p = q = 2$, dann erhalten wir die Räume $H^{s,s'}_{x_0}(\mathbb{R}^n)$, welche von Bony in [Bo84] betrachtet wurden. Der Fall $p = q = \infty$ ergibt die 2-mikrolokalen Räume $C^{s,s'}_{x_0}(\mathbb{R}^n)$ eingeführt von Jaffard in [Ja91] und ausgiebig behandelt von Meyer, Jaffard und Lévy-Vehel, Seuret in [JaMey96] und [LVSeu04].

Der mehr verallgemeinerte Fall $1 \le p, q \le \infty$, und Charakterisierungen von speziellen

Chirp Signalen sowie Beziehungen zu Gravitatioswellensignalen wurden von Xu und Meyer studiert [Xu96],[MeyXu97].
Desweiteren hat Andersson die Räume $C^{s,s'}_{x_0}(\mathbb{R}^n)$ auf die Skala der Triebel-Lizorkin Räume $F^{s,s'}_{pq}(\mathbb{R}^n, x_0)$ für $1 \leq p, q \leq \infty$ ausgeweitet und gab in [And97] lokale Charakterisierungen dieser Räume. Eine Verallgemeinerung in Bezug auf die Gewichtsfolge sind die Räume $B^{s,s'}_{pq}(\mathbb{R}^n, U)$, wobei die Gewichte

$$w_j(x) = (1 + 2^j \operatorname{dist}(x, U))^{s'}$$

für eine offene Teilmenge $U \subset \mathbb{R}^n$ und $s' \in \mathbb{R}$ erfüllen. Diese wurden von Moritoh und Yamada in [MoYa04] abgehandelt.
Bis auf die Räume von Andersson in der Triebel-Lizorkin Skala sind alle oben erwähnten Verallgemeinerungen der 2-mikrolokalen Räume $C^{s,s'}_{x_0}(\mathbb{R}^n)$, von Bony und Jaffard, Spezialfälle der Definition von $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$.

Man kann $\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)$ folgendermaßen umschreiben

$$[\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)](x) = (2\pi)^{n/2}[(\mathcal{F}^{-1}\varphi_j) * f](x) \ . \tag{0.3}$$

Die Funktionen $\mathcal{F}^{-1}\varphi_j$ haben keinen kompakten Träger. Das heißt, um $\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)$ in $x \in \mathbb{R}^n$ zu berechnen, brauchen wir f global. Wir werden zeigen, dass die Funktionen $\mathcal{F}^{-1}\varphi_j$ in (0.3) und (0.1), (0.2) durch glatte Funktionen, mit kompakten Träger in einer Kugel vom Radius $c2^{-j}$, ersetzt werden können. Dies führt uns schließlich zu lokalen Charakterisierungen der Räume $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$. Solche lokalen Charakterisierungen sind wohlbekannt für gewichtete und ungewichtete Besovräume (siehe [Tri92] und [Tri06]) und haben sich als äußerst nützlich erwiesen einige Schlüsselprobleme, wie punktweise Multiplikatoren und die Invarianz unter Diffeomorphismen, zu beweisen. Weiterhin ebnet die lokale Charakterisierung den Weg zur Wavelet Zerlegung und weiteren Diskritisierungen (siehe [Tri06] für die klassischen Besovräume).
Dazu seien $\{a_{\nu m}\} \subset C^k(\mathbb{R}^n)$ bestimmte Bausteine (Atome oder Moleküle), dann finden wir für gegebenes $f \in \mathcal{S}'(\mathbb{R}^n)$ Koeffizienten $\{\lambda_{\nu m}\} \subset \mathbb{C}$, so dass

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m} \quad \text{mit Konvergenz in } \mathcal{S}'(\mathbb{R}^n)$$

und

$$\big\| \lambda |\, b^{s,mloc}_{pq}(\boldsymbol{w}) \big\| = \left(\sum_{\nu=0}^{\infty} 2^{\nu(s-n/p)q} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p w^p_\nu(2^{-\nu} m) \right)^{q/p} \right)^{1/q}$$

eine äquivalente quasi-Norm auf $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ ist.
Die vorigen Resultate benutzen wir um die Wavelet Charakterisierung der 2-mikrolokalen Besovräume zu erhalten. Das Ergebnis fügt sich nahtlos in die bereits existierende Theorie for Besovräume ([Tri06]) und die Wavelet Charakterisierung von $C^{s,s'}_{x_0}(\mathbb{R}^n)$ ([Ja91]) ein. Mit Hilfe der Wavelet Charakterisierung können wir weitere Resultate beweisen,

wie die Invarianz von $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ unter Pseudodifferentialoperatoren der Ordnung 0 und die Verbindung der Räume $B^{s,s'}_{pq}(\mathbb{R}^n, U)$ zu den Räumen variierender Glattheit welche von Schneider in [Schn07] eingeführt und untersucht wurden.

Die vorliegende Arbeit richtet sich nach folgender Struktur. Das zweite Kapitel enthält alle grundlegenden Definitionen und einige Eigenschaften, wie die Unabhängigkeit der Räume von der Zerlegung der Eins $\{\varphi_j\}_{j\in\mathbb{N}_0}$ und die Lifteigenschaft. Dabei benutzen wir hauptsächlich ein Fouriersches Multiplikatortheorem für gewichtete ganz-analytische Funktionen aus [SchmTri87]. Anschließend beweisen wir Einbettungstheoreme in verschiedenen Skalen welche hauptsächlich auf einer gewichteten Nikols'kij Ungleichung [SchmTri87] aufbauen.
Das Herzstück der Arbeit sind Kapitel 3 und 5. Im dritten Kapitel beweisen wir die Charakterisierung der Räume durch die lokalen Mittel. Dazu nutzen wir Maximalfunktionen und Maximalungleichungen und folgen den Arbeiten von Triebel [Tri92], Rychkov [Ry99] und, in einem anderen Zusammenhang, Vybiral [Vyb06].
Im nächsten Kapitel verwenden wir die Resultate von Kapitel 3 (lokale Mittel) um ein Theorem zu punktweiser Multiplikation und die Invarianz der Räume unter einer speziellen Klasse von Diffeomorphismen zu beweisen.
Das nächste Hauptkapitel, Kapitel 5, ist den Charakterisierungen von $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ mit Hilfe von Zerlegungen gewidmet. Die Zerlegungen in Atome, Moleküle und Wavelets im Sinne von [FrJa85] und [Kyr03] werden hier angegeben.
Das sechste Kapitel dient dazu die Folgerungen aus der Waveletzerlegung darzulegen. Wir erhalten die Invarianz der Räume unter Pseudodifferrentialoperatoren, können die Schärfe der Einbettungen aus Kapitel 2 beweisen und zeigen die Verbindung zu den Räumen variierender Glattheit [Schn07] auf.
Das letzte Kapitel gibt einen Überblick zu der Theorie der lokalen Räume $B^{s,s'}_{pq}(U)^{loc}$. Diese Räume sind das angemessene Werkzeug, um punktweise Regularität von Funktionen zu untersuchen. Wir geben einige grundlegende Resultate um die verallgemeinerten 2-mikrolokalen Räume $B^{s,s'}_{pq}(U)^{loc}$ mit der bekannten Theorie, im Spezialfall $U = \{x_0\}$ und $p = q = \infty$ in [JaMey96] und [LVSeu04] für $C^{s,s'}_{x_0}(\mathbb{R}^n)^{loc}$, zu verknüpfen.

1 Introduction

The concept of 2-microlocal analysis or 2-microlocal function spaces is due to Jean-Michel Bony [Bo84] and Stéphane Jaffard [Ja91]. It is an appropriate instrument to describe the local regularity and the oscillatory behavior of functions near to singularities.
The approach is Fourier-analytical using Littlewood-Paley-analysis of distributions. To study such spaces was firstly suggested by Jaak Peetre in [Pe75] on page 266. The theory has been elaborated and widely used, in special cases, in fractal analysis and signal processing by several authors. We refer to [Ja91], [JaMey96], [LVSeu04], [Mey97], [MeyXu97], [MoYa04] and [Xu96].
The main achievements are closely related to the use of wavelet methods and, as a consequence, wavelet characterizations of 2-microlocal spaces. Here, we intend to give a unified Fourier-analytical approach to generalize 2-microlocal Besov spaces and we are interested in local and wavelet characterizations of the spaces under consideration.
Therefore, let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ be a smooth resolution of unity (see Section 2.2 for the precise definition) and let $\{w_j\}_{j\in\mathbb{N}_0}$ be a sequence of weight functions satisfying

$$\begin{aligned} &0 < w_j(x) \le C w_j(y)(1+2^j|x-y|)^\alpha \\ &2^{-\alpha_1} w_j(x) \le w_{j+1}(x) \le 2^{\alpha_2} w_j(x) \ , \end{aligned}$$

for $x, y \in \mathbb{R}^n$, $j \in \mathbb{N}_0$ and $\alpha, \alpha_1, \alpha_2 \ge 0$. $\mathcal{F}$ and $\mathcal{F}^{-1}$ stand for the Fourier transform and its inverse in the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions, respectively. Let $0 < p, q \le \infty$ and $s \in \mathbb{R}$. Then we introduce $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ as the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\left\| f \,|\, B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| w_j \mathcal{F}^{-1}(\varphi_j \mathcal{F} f) \,\middle|\, L_p(\mathbb{R}^n) \right\|^q \right)^{1/q} < \infty \tag{1.1}$$

for $0 < q < \infty$ and

$$\left\| f \,|\, B^{s,mloc}_{p\infty}(\mathbb{R}^n, \boldsymbol{w}) \right\| = \sup_{j\in\mathbb{N}_0} 2^{js} \left\| w_j \mathcal{F}^{-1}(\varphi_j \mathcal{F} f) \,\middle|\, L_p(\mathbb{R}^n) \right\| < \infty \ , \tag{1.2}$$

for $q = \infty$. As a special case, let $w_j(x) = (1+2^j|x-x_0|)^{s'}$ for $s' \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$ and $j \in \mathbb{N}_0$. If $p = q = 2$ we obtain the spaces $H^{s,s'}_{x_0}(\mathbb{R}^n)$ considered by Bony in [Bo84]. The case $p = q = \infty$ yields the 2-microlocal spaces $C^{s,s'}_{x_0}(\mathbb{R}^n)$ introduced by Jaffard in [Ja91] and extensively treated by Meyer, Jaffard and Lévy-Vehel, Seuret ([JaMey96],[LVSeu04]).
The more general case $1 \le p, q \le \infty$, and characterizations of chirp-like signals as well as relations to gravitational wave signals, have been studied by Xu and Meyer ([Xu96],[MeyXu97]).
Moreover, Andersson has generalized the spaces $C^{s,s'}_{x_0}(\mathbb{R}^n)$ to the scale of Triebel-Lizorkin spaces $F^{s,s'}_{pq}(\mathbb{R}^n, x_0)$ for $1 \le p, q \le \infty$ and has given in [And97] local characterizations of

this spaces. A generalization with respect to the weight sequence, the spaces $B^{s,s'}_{pq}(\mathbb{R}^n, U)$, was examined by Moritoh and Yamada in [MoYa04]. The weights in this work have to satisfy

$$w_j(x) = (1 + 2^j \operatorname{dist}(x, U))^{s'}$$

for an open subset $U \subset \mathbb{R}^n$ and $s' \in \mathbb{R}$.
With exception of the spaces of Andersson, all above mentioned spaces do fit in the scale of $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ and are a generalization of the original spaces $C^{s,s'}_{x_0}(\mathbb{R}^n)$ by Bony and Jaffard.

We can rewrite $\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)$ by

$$[\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)](x) = (2\pi)^{n/2}[(\mathcal{F}^{-1}\varphi_j) * f](x) \ . \tag{1.3}$$

The functions $\mathcal{F}^{-1}\varphi_j$ do not have compact support. In particular, to compute the building blocks $\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)$ in $x \in \mathbb{R}^n$ we need f globally. Roughly speaking, we shall show that the functions $\mathcal{F}^{-1}\varphi_j$ in (1.3) and (1.1), (1.2), respectively, can be replaced by smooth functions with compact support in a ball of radius $c2^{-j}$ (c is a constant). This leads to local characterizations of our spaces. Characterizations of such a type are well known for weighted and unweighted Besov spaces (see for instance [Tri92] and [Tri06]) and turned out to be very useful to solve some key problems as the behavior by pointwise multiplication and invariance under diffeomorphisms. Moreover, it paves the way to atomic and wavelet representations as well as to discretizations (see [Tri06] for classical Besov spaces) and isomorphisms to corresponding sequence spaces.
Let $\{a_{\nu m}\} \subset C^k(\mathbb{R}^n)$ be some building blocks (atoms or molecules), then for given $f \in \mathcal{S}'(\mathbb{R}^n)$ we can find coefficients $\{\lambda_{\nu m}\} \subset \mathbb{C}$ such that

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m} \quad \text{with convergence in } \mathcal{S}'(\mathbb{R}^n)$$

and

$$\big\| \lambda \big| \, b^{s,mloc}_{pq}(\boldsymbol{w}) \big\| = \left(\sum_{\nu=0}^{\infty} 2^{\nu(s-n/p)q} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p w^p_\nu(2^{-\nu} m) \right)^{q/p} \right)^{1/q}$$

is an equivalent quasi-norm on $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$.
Combining the previous results we get the wavelet characterization of the 2-microlocal Besov spaces and the result seamlessly incorporates in the known theory for Besov spaces [Tri06] and the wavelet characterization for $C^{s,s'}_{x_0}(\mathbb{R}^n)$ [Ja91]. With the wavelet characterization on hand, we are able to prove further results as the invariance of $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ under pseudodifferential operators and the connection to the spaces of varying smoothness by Schneider in [Schn07].

This work is organized as follows. Chapter 2 contains all definitions and some basic

properties such as the independence of $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ of the choice of the resolution of unity $\{\varphi_j\}_{j\in\mathbb{N}_0}$ and the lift property. Here we rely on Fourier multiplier theorems for weighted spaces of entire analytic functions which can be found in [SchmTri87]. Then we deal with embedding theorems for different metrics based on weighted Nikols'kij inequalities ([SchmTri87]).
The main parts are Chapters 3 and 5. In Chapter 3 we give the characterization by local means. We use maximal functions and inequalities and follow ideas in [Tri92], [Ry99] and [Vyb06] in a different context.
In the next chapter we apply the results of Chapter 3 (local means) to prove a theorem on pointwise multiplication. Finally, we use the local means characterization to prove that the spaces $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ are invariant under a special class of diffeomorphisms.
Chapter 5 is devoted to the study of characterizations of $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ by decompositions. We give the atomic, molecular and wavelet decomposition in the spirit of [FrJa85] and [Kyr03].
In Chapter 6 we apply the wavelet characterization to get the invariance of $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ by pseudodifferential operators of order 0 (as in [Ja91]), to prove that the embeddings from Section 2.5.1 are sharp and to give a connection of the 2-microlocal Besov spaces $B^{s,s'}_{pq}(\mathbb{R}^n, U)$ to the spaces of varying smoothness from Schneider ([Schn07]).
In the last chapter we give a short outline on the local spaces $B^{s,s'}_{pq}(U)^{loc}$. These spaces are an appropriate tool to measure local smoothness of functions. Some fundamental results are given to connect the spaces $B^{s,s'}_{pq}(U)^{loc}$ to the known theory, in the special case $U = \{x_0\}$ and $p = q = \infty$, of [JaMey96] and [LVSeu04] for $C^{s,s'}_{x_0}(\mathbb{R}^n)^{loc}$.

2 The 2-microlocal Besov spaces $\mathbf{B^{s,mloc}_{pq}}(\mathbb{R}^\mathbf{n}, \boldsymbol{w})$

2.1 Preliminaries

As usual $\mathbb{R}^n$ symbolizes the $n-$dimensional Euclidean space, $\mathbb{N}$ is the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathbb{Z}$ and $\mathbb{C}$ stand for the sets of integers and complex numbers, respectively.
The points of the Euclidean space $\mathbb{R}^n$ are denoted by $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n), \dots$.
If $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$ is a multi-index, then its length is denoted by $|\beta| = \sum_{j=1}^n \beta_j$.
The derivatives $D^\beta = \partial^{|\beta|}/\partial^{\beta_1} \cdots \partial^{\beta_n}$ have to be understood in the distributional sense.
We put $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$.
The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is the space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$. Its topology is generated by the norms

$$\|\varphi\|_{k,l} = \sup_{x \in \mathbb{R}^n} (1 + |x|)^k \sum_{|\beta| \le l} |D^\beta \varphi(x)| \ , \quad k, l \in \mathbb{N}_0 \ . \tag{2.1}$$

A linear mapping $f : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ is called a tempered distribution, if there are a constant $c > 0$ and $k, l \in \mathbb{N}_0$ such that

$$|f(\varphi)| \le c \|\varphi\|_{k,l}$$

holds for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. The collection of all such mappings is denoted by $\mathcal{S}'(\mathbb{R}^n)$. The Fourier transform is defined on both spaces $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ and is given by

$$(\mathcal{F}f)(\varphi) := f(\mathcal{F}\varphi) \ , \qquad \varphi \in \mathcal{S}(\mathbb{R}^n) \ , \quad f \in \mathcal{S}'(\mathbb{R}^n)$$

where

$$\mathcal{F}\varphi(\xi) := \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} \mathrm{e}^{-ix\cdot\xi} \, \varphi(x) dx \ .$$

Here $x \cdot \xi = x_1\xi_1 + \dots + x_n\xi_n$ stands for the inner product. The inverse Fourier transform is denoted by $\mathcal{F}^{-1}\varphi$ or $\varphi^\vee$ and we often write $\hat{\varphi}$ instead of $\mathcal{F}\varphi$.

Vector-valued sequence spaces

We say a vector space E is a quasi Banach space, if it is quasi-normed by $\|\cdot| E\|$ and complete. That means $\|\cdot| E\|$ fulfills the norm conditions but the triangle inequality changes to

$$\|x + y| E\| \le c \,\|x| E\| + c \,\|y| E\| \ , \tag{2.2}$$

for $c \geq 1$. If $c = 1$ in (2.2) then $\| \cdot | E\|$ is a norm and E is a Banach space.
As usual $L_p(\mathbb{R}^n)$ for $0 < p \leq \infty$ stands for the Lebesgue spaces on $\mathbb{R}^n$ normed by (quasi-normed for $p < 1$)

$$\| f | L_p(\mathbb{R}^n)\| = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p} \qquad \text{for } 0 < p < \infty \text{ and}$$
$$\| f | L_\infty(\mathbb{R}^n)\| = \operatorname*{ess-sup}_{x \in \mathbb{R}^n} |f(x)| \ .$$

If w is a positive measurable function on $\mathbb{R}^n$, we denote the weighted Lebesgue spaces by $L_p(\mathbb{R}^n, w)$ and the quasi-norms are defined for $0 < p \leq \infty$ by

$$\| f | L_p(\mathbb{R}^n, w)\| = \| wf | L_p(\mathbb{R}^n)\| = \left(\int_{\mathbb{R}^n} |f(x)|^p w^p(x) dx \right)^{1/p} ,$$

with the usual modification if $p = \infty$. For a complex-valued sequence $a = \{a_j\}_{j=0}^\infty$ the sequence spaces ℓ_q for $0 < q \leq \infty$ are normed (quasi-normed for $q < 1$) by

$$\| a | \ell_q\| = \left(\sum_{j=0}^\infty |a_j|^q \right)^{1/q} \qquad \text{for } 0 < q < \infty \text{ and}$$
$$\| a | \ell_\infty\| = \sup_{j \in \mathbb{N}_0} |a_j| \ .$$

Let $\{f_k\}_{k \in \mathbb{N}_0}$ be a sequence of complex-valued measurable functions, $0 < p \leq \infty$ and $0 < q \leq \infty$. Then we put

$$\| f_k(x) | \ell_q(L_p)\| = \| \{f_k\}_{k \in \mathbb{N}_0} | \ell_q(L_p)\| = \left(\sum_{k=0}^\infty \left(\int_{\mathbb{R}^n} |f_k(x)|^p dx \right)^{q/p} \right)^{1/q} = \left(\sum_{k=0}^\infty \| f_k | L_p\|^q \right)^{1/q},$$

also with the above modifications for $p = \infty$ or $q = \infty$.
The constant c adds up all unimportant constants. So the value of the constant c may change from one occurrence to another. By $a_k \sim b_k$ we mean that there are two constants $c_1, c_2 > 0$ such that $c_1 a_k \leq b_k \leq c_2 a_k$ for all admissible k.

2.2 Definitions and basic properties

In this section we present the Fourier analytical definition of generalized 2-microlocal Besov spaces $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ and we prove the basic properties in analogy to the classical Besov spaces. To this end we need smooth resolutions of unity and we introduce our admissible weight sequences $\boldsymbol{w} = \{w_j\}_{j \in \mathbb{N}_0}$.

Definition 2.1 (Admissible weight sequence)**:** *Let $\alpha, \alpha_1, \alpha_2 \geq 0$. We say that a sequence of non-negative measurable functions $\boldsymbol{w} = \{w_j\}_{j=0}^\infty$ belongs to the class $\mathcal{W}^\alpha_{\alpha_1,\alpha_2}$ if, and only if,*

(i) *There exists a constant* $\mathrm{C} > 0$ *such that*

$$0 < w_j(x) \le \mathrm{C} w_j(y) \left(1 + 2^j |x-y|\right)^{\alpha} \quad \textit{for all } j \in \mathbb{N}_0 \textit{ and all } x, y \in \mathbb{R}^n. \tag{2.3}$$

(ii) *For all* $j \in \mathbb{N}_0$ *we have*

$$2^{-\alpha_1} w_j(x) \le w_{j+1}(x) \le 2^{\alpha_2} w_j(x) \quad \textit{for all } x \in \mathbb{R}^n. \tag{2.4}$$

Such a system $\{w_j\}_{j=0}^{\infty} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ *is called* admissible weight sequence.

Remark 2.2: A non-negative measurable function ϱ is called an *admissible weight function* if there exist constants $\alpha_\varrho \ge 0$ and $\mathrm{C}_\varrho > 0$ such that

$$0 < \varrho(x) \le \mathrm{C}_\varrho \varrho(y)(1 + |x-y|)^{\alpha_\varrho} \quad \text{holds for every } x, y \in \mathbb{R}^n. \tag{2.5}$$

If $\boldsymbol{w} = \{w_j\}_{j=0}^{\infty}$ is an admissible weight sequence, each w_j is an admissible weight function, but in general the constant C_{w_j} depends on $j \in \mathbb{N}_0$.

Remark 2.3: If we use $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ without any restrictions, then $\alpha, \alpha_1, \alpha_2 \ge 0$ are arbitrary but fixed numbers.

Remark 2.4: If $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $\widetilde{\boldsymbol{w}} \in \mathcal{W}^{\beta}_{\beta_1,\beta_2}$ and $\lambda > 0$, it is easy to check:

(a) The sequence $\boldsymbol{w^{-1}} = \{w_j^{-1}\}_{j=0}^{\infty}$ belongs to the class $\mathcal{W}^{\alpha}_{\alpha_2,\alpha_1}$.

(b) The sequence $\lambda \boldsymbol{w}$ belongs to the class $\mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$.

(c) The sequence $\boldsymbol{w^{\lambda}} = \{w_j^{\lambda}\}_{j=0}^{\infty}$ belongs to the class $\mathcal{W}^{\lambda\alpha}_{\lambda\alpha_1,\lambda\alpha_2}$.

(d) The sequence $\boldsymbol{w} + \widetilde{\boldsymbol{w}}$ belongs to the class $\mathcal{W}^{\max(\alpha,\beta)}_{\max(\alpha_1,\beta_1),\max(\alpha_2,\beta_2)}$.

(e) The sequence $\boldsymbol{w} \cdot \widetilde{\boldsymbol{w}}$ belongs to the class $\mathcal{W}^{\alpha+\beta}_{\alpha_1+\beta_1,\alpha_2+\beta_2}$.

Example 2.5: Let $U \neq \emptyset$ be a subset of $\mathbb{R}^n$. We denote by $\operatorname{dist}(x, U) = \inf_{z \in U} |x - z|$ the distance of $x \in \mathbb{R}^n$ from U. A typical admissible weight sequence for fixed $U \subset \mathbb{R}^n$ and $s' \in \mathbb{R}$ is given by

$$w_j(x) := \left(1 + 2^j \operatorname{dist}(x, U)\right)^{s'} \quad \text{for } j \in \mathbb{N}_0. \tag{2.6}$$

We have for $s' \ge 0$

$$w_j(x) \le w_{j+1}(x) \le 2^{s'} w_j(x) \quad \text{and for } s' < 0 \quad 2^{s'} w_j(x) \le w_{j+1}(x) \le w_j(x)\ .$$

Hence, for all $j \in \mathbb{N}_0$ and all fixed $s' \in \mathbb{R}$

$$2^{-\max(0,-s')} w_j(x) \le w_{j+1}(x) \le 2^{\max(0,s')} w_j(x) \qquad \text{for every } x \in \mathbb{R}^n. \tag{2.7}$$

From the inequality $\operatorname{dist}(x, U) \le |x - y| + \operatorname{dist}(y, U)$ we derive for $s' \ge 0$

$$\begin{aligned} w_j(x) &= \left(1 + 2^j \operatorname{dist}(x, U)\right)^{s'} \\ &\le \left(1 + 2^j |x-y| + 2^j \operatorname{dist}(y, U)\right)^{s'}\ . \end{aligned}$$

Since $a+b \leq 2ab$ for $a, b \geq 1$, we get

$$\begin{aligned} w_j(x) &\leq \left[2\left(1+2^j \operatorname{dist}(y,U)\right)(1+2^j|x-y|)\right]^{s'} \\ &= 2^{s'} w_j(y)(1+2^j|x-y|)^{s'} , \end{aligned}$$

for all $x, y \in \mathbb{R}^n$ and all $j \in \mathbb{N}_0$. If $s' < 0$ we can do the same calculation for the inverse weight sequence $\boldsymbol{w}^{-1}$ and according to Remark 2.4(a) we obtain

$$w_j(x) \leq 2^{-s'} w_j(y)(1+2^j|x-y|)^{-s'} ,$$

for all $x, y \in \mathbb{R}^n$ and all $j \in \mathbb{N}_0$. Finally, we have for fixed $s' \in \mathbb{R}$ and all $j \in \mathbb{N}_0$

$$0 < w_j(x) \leq 2^{|s'|} w_j(y)(1+2^j|x-y|)^{|s'|} , \tag{2.8}$$

for all $x, y \in \mathbb{R}^n$. Together with (2.7) we get $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ if $|s'| \leq \alpha$, $\max(0,-s') \leq \alpha_1$ and $\max(0,s') \leq \alpha_2$.
A special case is $U = \{x_0\}$ for $x_0 \in \mathbb{R}^n$. Then $\operatorname{dist}(U,x) = |x-x_0|$ and we get the well known 2-microlocal weights [JaMey96]:

$$w_j(x) = (1+2^j|x-x_0|)^{s'} \qquad \text{for } j \in \mathbb{N}_0. \tag{2.9}$$

If U is an open subset of $\mathbb{R}^n$, then (2.6) gives us the weight sequence Moritoh and Yamada used in [MoYa04].

Example 2.6: Let $w : \mathbb{R}^n \to [0,\infty)$ be a measurable function with the properties: There are constants $\mathcal{C}_1$, $\mathcal{C}_2 \geq 1$ and $\beta \geq 1$ such that for all $x, y \in \mathbb{R}^n$

$$0 \leq w(x) \leq \mathcal{C}_1 w(y) + \mathcal{C}_2 |x-y|^{\beta} . \tag{2.10}$$

For fixed $s' \in \mathbb{R}$ and all $j \in \mathbb{N}_0$ we define

$$w_j(x) = \left(1+2^j w(x)\right)^{s'/\beta} \quad \text{for all } x \in \mathbb{R}^n. \tag{2.11}$$

By analogy to Example 2.5 above we get

$$\begin{aligned} 0 < w_j(x) \leq (2\,\mathcal{C}_1\mathcal{C}_2)^{|s'|}\, w_j(y)\left(1+2^j|x-y|\right)^{|s'|} \quad &\text{and} \\ 2^{-\max(0,-s')} w_j(x) \leq w_{j+1}(x) \leq 2^{\max(0,s')} w_j(x) \quad &\text{holds for all } x, y \in \mathbb{R}^n \text{ and } j \in \mathbb{N}_0. \end{aligned}$$

Hence, we have $\boldsymbol{w} = \{w_j\}_{j\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ for all $\alpha \geq |s'|$ and $\alpha_1 \geq \max(0,-s')$, $\alpha_2 \geq \max(0,s')$.
As a special case we choose $w : \mathbb{R}^n \to [0,\infty)$ subadditive that is

$$\begin{aligned} 0 \leq w(x+y) \leq \tilde{c}_1 \left(w(x)+w(y)\right) \quad &\text{and in addition we need} \\ w(x) \leq \tilde{c}_2 |x|^{\beta} \quad &\text{for all } x \in \mathbb{R}^n \text{ and fixed } \tilde{c}_1,\, \tilde{c}_2,\, \beta \geq 1. \end{aligned}$$

Thus we have (2.10) with $\mathcal{C}_1 = \tilde{c}_1$ and $\mathcal{C}_2 = \tilde{c}_1\tilde{c}_2$ and we can define the admissible weight sequence as in (2.11).

Next we define the resolution of unity.

Definition 2.7 (Resolution of unity)**:** *A system* $\boldsymbol{\varphi} = \{\varphi_j\}_{j=0}^{\infty} \subset \mathcal{S}(\mathbb{R}^n)$ *belongs to the class* $\Phi(\mathbb{R}^n)$ *if, and only if,*

(i) $\operatorname{supp}\varphi_0 \subseteq \{x \in \mathbb{R}^n : |x| \le 2\}$ *and* $\operatorname{supp}\varphi_j \subseteq \{x \in \mathbb{R}^n : 2^{j-1} \le |x| \le 2^{j+1}\}$*;*

(ii) For each $\beta \in \mathbb{N}_0^n$ *there exist constants* $c_\beta > 0$ *such that*

$$2^{j|\beta|} \sup_{x\in\mathbb{R}^n} |D^\beta \varphi_j(x)| \le c_\beta \quad \textit{holds for all } j \in \mathbb{N}_0.$$

(iii) For all $x \in \mathbb{R}^n$ *we have*

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1 \ .$$

Remark 2.8: Such a resolution of unity can easily be constructed. Consider the following example. Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ with $\varphi_0(x) = 1$ for $|x| \le 1$ and $\operatorname{supp}\varphi_0 \subseteq \{x \in \mathbb{R}^n : |x| \le 2\}$. For $j \ge 1$ we define

$$\varphi_j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x) \ .$$

Now it is obvious that $\boldsymbol{\varphi} = \{\varphi_j\}_{j\in\mathbb{N}_0} \in \Phi(\mathbb{R}^n)$.

Definition 2.9: *Let* $\{\varphi_j\}_{j\in\mathbb{N}_0} \in \Phi(\mathbb{R}^n)$ *be a resolution of unity and let* $\boldsymbol{w} = \{w_j\}_{j\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$. *Further, let* $0 < p \le \infty$, $0 < q \le \infty$ *and* $s \in \mathbb{R}$. *Then we define*

$$B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) = \left\{ f \in S' : \left\| f \middle| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right\|_{\varphi} < \infty \right\} \quad , \textit{ where}$$

$$\left\| f \middle| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right\|_{\varphi} = \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| w_j (\varphi_j \hat{f})^\vee \middle| L_p(\mathbb{R}^n) \right\|^q \right)^{1/q} \quad ,$$

with the usual modifications if p or q are equal to infinity.

Remark 2.10: One recognizes immediately that for $w_j \equiv 1$ one obtains the usual Besov spaces $B^s_{pq}(\mathbb{R}^n)$ (see [Tri83]), which are given by the norms

$$\left\| f \middle| B^s_{pq}(\mathbb{R}^n) \right\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| (\varphi_j \hat{f})^\vee \middle| L_p(\mathbb{R}^n) \right\|^q \right)^{1/q} \quad . \tag{2.12}$$

If one defines the admissible weight sequence as $w_j(x) = \varrho(x)$ for each $j \in \mathbb{N}_0$ and ϱ fulfills (2.5), we obtain the usual weighted Besov spaces $B^s_{pq}(\mathbb{R}^n, \varrho)$, see [EdTri96, Chapter 4]. In particular, if $w_j(x) = (1 + |x|^2)^{\alpha/2}$ then we denote this weighted Besov spaces by $B^s_{pq}(\mathbb{R}^n, \alpha)$ (see Section 2.5.1).

Firstly, we have to prove that Definition 2.9 is independent of the chosen system $\{\varphi_j\}_{j\in\mathbb{N}_0} \in \Phi(\mathbb{R}^n)$. We need a Fourier multiplier theorem for weighted Lebesgue spaces of entire analytic functions as in [SchmTri87]. We define the Sobolev spaces $W_2^\kappa(\mathbb{R}^n)$ for $\kappa \in \mathbb{N}_0$. A function $f \in L_2(\mathbb{R}^n)$ belongs to $W_2^\kappa(\mathbb{R}^n)$ if

$$\| f | \, W_2^\kappa(\mathbb{R}^n)\| := \left(\sum_{|\gamma|\leq\kappa} \| D^\gamma f | \, L_2(\mathbb{R}^n)\|^2\right)^{1/2} < \infty \; . \tag{2.13}$$

With $B_r(x)$ we denote the closed Ball centered at $x \in \mathbb{R}^n$ and with radius $r > 0$. We write $\mathcal{F}^{-1}M\mathcal{F}f$ instead of $\mathcal{F}^{-1}[M(\mathcal{F}f)]$.

Theorem 2.11 ([SchmTri87], Remark 4/1.7.5)**:** *Let $\varrho : \mathbb{R}^n \to \mathbb{R}$ be an admissible weight which satisfies* (2.5) *for some $\alpha_\varrho \geq 0$. Furthermore, let $B_b(0) = \{y \in \mathbb{R}^n : |y| \leq b\}$ for $b > 0$ and let $0 < p \leq \infty$. Then for every $\kappa \in \mathbb{N}$ with*

$$\kappa > n\left(\frac{1}{\min(1,p)} - \frac{1}{2}\right) + \alpha_\varrho \tag{2.14}$$

there exists a constant $c > 0$ (depending on b) such that

$$\left\| \varrho\mathcal{F}^{-1}M\mathcal{F}f \right| L_p(\mathbb{R}^n)\right\| \leq c \, \| M | \, W_2^\kappa(\mathbb{R}^n)\| \, \| \varrho f | \, L_p(\mathbb{R}^n)\| \tag{2.15}$$

holds for all $f \in L_p(\mathbb{R}^n, \varrho) \cap \mathcal{S}'(\mathbb{R}^n)$ with $\operatorname{supp}\mathcal{F}f \subseteq B_b(0)$ *and all $M \in \mathcal{S}(\mathbb{R}^n)$.*

Remark 2.12: Additionally, we need a corollary of Theorem 2.11. We assume that the weight satisfies

$$0 < \varrho(x) \leq \mathrm{C}_\varrho \varrho(y) \, (1 + ab|x-y|)^{\alpha_\varrho} \quad \text{for fixed } a > 0 \text{ and all } x, y \in \mathbb{R}^n. \tag{2.16}$$

If $f \in L_p(\mathbb{R}^n, \varrho)$ with $\operatorname{supp}\mathcal{F}f \subset B_b(0)$, then $\operatorname{supp}\mathcal{F}[f(b^{-1}x)] \subset B_1(0)$ and by the properties of the Fourier transform

$$\left(\varrho\mathcal{F}^{-1}M\mathcal{F}f\right)(x) = \left\{\varrho(b^{-1}\cdot)\mathcal{F}^{-1}\left[M(b\cdot)\left(\mathcal{F}f(b^{-1}\cdot)\right)(\cdot)\right]\right\}(bx) \; . \tag{2.17}$$

Therefore, we obtain

$$\begin{aligned}\left\| \left(\varrho\mathcal{F}^{-1}M\mathcal{F}f\right)(x) \right| L_p(\mathbb{R}^n)\right\| &= \left\| \left\{\varrho(b^{-1}\cdot)\mathcal{F}^{-1}\left[M(b\cdot)\left(\mathcal{F}f(b^{-1}\cdot)\right)(\cdot)\right]\right\}(bx) \right| L_p(\mathbb{R}^n)\right\| \\ &= b^{-\frac{n}{p}} \left\| \left\{\varrho(b^{-1}\cdot)\mathcal{F}^{-1}\left[M(b\cdot)\left(\mathcal{F}f(b^{-1}\cdot)\right)(\cdot)\right]\right\}(x) \right| L_p(\mathbb{R}^n)\right\| \; .\end{aligned}$$

For the weight function $r(x) = \varrho(b^{-1}x)$ we have $\alpha_r = \alpha_\varrho$ and

$$0 < r(x) \leq \max(1,a)^{\alpha_\varrho}\mathrm{C}_\varrho \, r(y) \, (1 + |x-y|)^{\alpha_\varrho} = \mathrm{C}'_\varrho \, r(y) \, (1 + |x-y|)^{\alpha_\varrho} \; .$$

We can apply Theorem 2.11 and obtain

$$\begin{aligned}\left\| \left(\varrho\mathcal{F}^{-1}M\mathcal{F}f\right)(x) \right| L_p(\mathbb{R}^n)\right\| &\leq c \cdot \mathrm{C}'_\varrho b^{-\frac{n}{p}} \, \| M(b\cdot) | \, W_2^\kappa(\mathbb{R}^n)\| \left\| \varrho(b^{-1}\cdot)f(b^{-1}\cdot) \right| L_p(\mathbb{R}^n)\right\| \\ &= c\mathrm{C}_\varrho \, \| M(b\cdot) | \, W_2^\kappa(\mathbb{R}^n)\| \, \| \varrho f | \, L_p(\mathbb{R}^n)\| \end{aligned} \tag{2.18}$$

for $\kappa > n\left(\frac{1}{\min(1,p)} - \frac{1}{2}\right) + \alpha_\varrho$ and the constant $c > 0$ is independent of b.

Now, we are ready to show that Definition 2.9 of the spaces $B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})$ is independent of the chosen resolution of unity $\boldsymbol{\varphi} \in \Phi(\mathbb{R}^n)$.

Theorem 2.13 (Independence of the resolution of unity)**:** *Let $\boldsymbol{\varphi} = \{\varphi_j\}_{j\in\mathbb{N}_0}$, $\boldsymbol{\phi} = \{\phi_j\}_{j\in\mathbb{N}_0} \in \Phi(\mathbb{R}^n)$ be two resolutions of unity and let $\boldsymbol{w} = \{w_j\}_{j\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ be an admissible weight sequence. If $0 < p \le \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$, then we have*

$$\left\| f \,|\, B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\right\|_{\varphi} \sim \left\| f \,|\, B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\right\|_{\phi} \quad \textit{for all } f \in \mathcal{S}'(\mathbb{R}^n).$$

Proof: It is sufficient to show that there is a $c > 0$ such that for all $f \in \mathcal{S}'(\mathbb{R}^n)$ we have $\left\| f \,|\, B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\right\|_{\phi} \le c \left\| f \,|\, B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\right\|_{\varphi}$. Interchanging the roles of φ and ϕ we obtain the desired result.
Putting $\varphi_{-1} = 0$ we see

$$\phi_j(x) = \phi_j(x) \sum_{k=-1}^{1} \varphi_{j+k}(x) \quad \text{for all } j \in \mathbb{N}_0.$$

By the properties of the Fourier transform

$$w_j\{\mathcal{F}^{-1}\phi_j(\mathcal{F}f)\} = \sum_{k=-1}^{1} w_j \left\{\mathcal{F}^{-1}\phi_j\left(\mathcal{F}\left[\mathcal{F}^{-1}\varphi_{j+k}\left(\mathcal{F}f\right)\right]\right)\right\} .$$

Now, we apply (2.18) with $b = 2^{j+2}$, $M = \phi_j$ and $f = \mathcal{F}^{-1}\varphi_{j+k}\left(\mathcal{F}f\right)$ for $k \in \{-1,0,1\}$.
We get for every $j \in \mathbb{N}_0$

$$\begin{aligned} &\left\| w_j \left\{\mathcal{F}^{-1}\phi_j\left(\mathcal{F}\left[\mathcal{F}^{-1}\varphi_{j+k}\left(\mathcal{F}f\right)\right]\right)\right\} \,\middle|\, L_p(\mathbb{R}^n)\right\| \\ &\qquad \le c \left\| \phi_j(2^{j+2}\cdot) \,\middle|\, W^{\kappa}_2(\mathbb{R}^n)\right\| \left\| w_j \left\{\mathcal{F}^{-1}\varphi_{j+k}\left(\mathcal{F}f\right)\right\} \,\middle|\, L_p(\mathbb{R}^n)\right\| , \end{aligned} \tag{2.19}$$

with $\kappa > n\left(\frac{1}{\min(1,p)} - \frac{1}{2}\right) + \alpha$. By (2.3) and formula (2.18) the constant c does not depend on $j \in \mathbb{N}$. Since $\operatorname{supp}\phi_j(2^{j+2}\cdot) \subseteq B_1$ and using the properties of the resolution of unity, we have

$$\sup_{l\in\mathbb{N}_0} \left\| \phi_l(2^{l+2}\cdot) \,\middle|\, W^{\kappa}_2(\mathbb{R}^n)\right\| \le c \sup_{l\in\mathbb{N}_0}\sup_{|\beta|\le\kappa}\sup_{x\in\mathbb{R}^n} 2^{l|\beta|} |(D^{\beta}\phi_l)(x)| < c_{\kappa} .$$

We conclude that

$$\begin{aligned} \left\| w_j\{\mathcal{F}^{-1}\phi_j(\mathcal{F}f)\} \,\middle|\, L_p(\mathbb{R}^n)\right\| &\le c \sum_{k=-1}^{1} \left\| w_j \left\{\mathcal{F}^{-1}\phi_j\left(\mathcal{F}\left[\mathcal{F}^{-1}\varphi_{j+k}\left(\mathcal{F}f\right)\right]\right)\right\} \,\middle|\, L_p(\mathbb{R}^n)\right\| \\ &\le c' \sum_{k=-1}^{1} \left\| w_j \left\{\mathcal{F}^{-1}\varphi_{j+k}\left(\mathcal{F}f\right)\right\} \,\middle|\, L_p(\mathbb{R}^n)\right\| . \end{aligned}$$

Finally, multiplying by 2^{js}, using the property (2.4) of the admissible weight sequence and taking the ℓ_q quasi-norm with respect to j, we see that

$$\left(\sum_{j=0}^{\infty} 2^{jsq} \left\| w_j\{\mathcal{F}^{-1}\phi_j(\mathcal{F}f)\} \,\middle|\, L_p(\mathbb{R}^n)\right\|^q\right)^{1/q} \le c'(2^{s+\alpha_2} + 1 + 2^{-s+\alpha_1}) \left\| f \,|\, B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\right\|_{\varphi} .$$

This completes the proof. □

Remark 2.14: As in Theorem 2.3.3 in [Tri83] we can prove that $B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})$ is a quasi-Banach space for all $s\in\mathbb{R}$ and $0<p,q\le\infty$ and even a Banach space in the case $p,q\ge 1$.

2.3 Fourier multipliers

For convenience, we reformulate the above Fourier multiplier theorem for weighted Lebesgue spaces of entire analytic functions to get a Fourier multiplier theorem for the spaces $B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})$.

Definition 2.15: *Let $m(x)$ be a complex-valued infinitely differentiable function of at most polynomial growth on $\mathbb{R}^n$. Further, let $w\in\mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $0<p,q\le\infty$ and $s\in\mathbb{R}$. Then $m(x)$ is said to be a Fourier multiplier for $B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})$ if there exists a constant $c>0$ such that*

$$\left\|\mathcal{F}^{-1}m\mathcal{F}f\right| B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\big\|\le c\left\|f\right| B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\big\|$$

holds for all $f\in B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})$.

Now, we want to state a criterion for Fourier multipliers on $B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})$. To this end, we introduce some special functions $\{\phi_j\}_{j\in\mathbb{N}_0}\subset\mathcal{S}(\mathbb{R}^n)$ which fulfill $0\le\phi_j(x)\le 1$ and

$$\phi_0(x)=1 \quad \text{for } |x|\le 2 \text{ and } \operatorname{supp}\phi_0\subset\{x\in\mathbb{R}^n:|x|\le 4\}\ , \tag{2.20}$$

$$\phi_j(x)=\phi(2^{-j}x), \text{ with} \tag{2.21}$$

$$\phi(x)=1 \quad \text{for } \frac{1}{2}\le|x|\le 2 \text{ and } \operatorname{supp}\phi\subset\{x\in\mathbb{R}^n:\frac{1}{4}\le|x|\le 4\}\ . \tag{2.22}$$

Theorem 2.16: *Let $m(x)$ be a complex-valued infinitely differentiable function of at most polynomial growth on $\mathbb{R}^n$. Further, let $\boldsymbol{w}\in\mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $0<p,q\le\infty$, $s\in\mathbb{R}$ and let $\{\phi_j\}_{j\in\mathbb{N}_0}\subset\mathcal{S}(\mathbb{R}^n)$ satisfying (2.20)-(2.22). For $\kappa\in\mathbb{N}$ we define*

$$M_\kappa=\sup_{j\in\mathbb{N}_0}\left\|\phi_j(2^jx)m(2^jx)\right| W^{\kappa}_2(\mathbb{R}^n)\big\|\ . \tag{2.23}$$

Then for

$$\kappa>n\left(\frac{1}{\min(1,p)}-\frac{1}{2}\right)+\alpha$$

there exists a constant $c>0$ such that

$$\left\|\mathcal{F}^{-1}m\mathcal{F}f\right| B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\big\|\le cM_\kappa\left\|f\right| B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\big\|$$

holds for all $f\in B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})$.

Proof: Let $\{\varphi_j\}_{j\in\mathbb{N}_0}\subset\Phi(\mathbb{R}^n)$ be a fixed resolution of unity. Then we have for every $j\in\mathbb{N}_0$

$$\begin{aligned}\mathcal{F}^{-1}\varphi_j\mathcal{F}(\mathcal{F}^{-1}m\mathcal{F}f)&=\mathcal{F}^{-1}\varphi_j\phi_jm\mathcal{F}f\\&=\mathcal{F}^{-1}m_j\mathcal{F}f_j\ ,\end{aligned}$$

where $m_j(x) = \phi_j(x)m(x)$ and $f_j(x) = \mathcal{F}^{-1}\varphi_j\mathcal{F}f(x)$.
We show that there exists a constant $c > 0$ such that for all $j \in \mathbb{N}_0$

$$\left\| w_j\mathcal{F}^{-1}\phi_j m\mathcal{F}(\mathcal{F}^{-1}\varphi_j\mathcal{F}f)\right| L_p(\mathbb{R}^n)\right\| \le cM_\kappa \left\| w_j\mathcal{F}^{-1}\varphi_j\mathcal{F}f\right| L_p(\mathbb{R}^n)\right\| \ . \tag{2.24}$$

First of all, we prove (2.24) for the case of fixed $j \ge 1$ and we use (2.18). We know that $\mathrm{supp}(\mathcal{F}f_j) \subset B_{2^{j+2}}(0)$. Using (2.18) with $b = 2^{j+2}$ we get

$$\begin{aligned}\left\| \left(w_j\mathcal{F}^{-1}m_j\mathcal{F}f_j\right)(x)\right| L_p(\mathbb{R}^n)\right\| &\le c\mathrm{C}_w \left\| m_j(2^{j+2}\cdot)\right| W_2^\kappa(\mathbb{R}^n)\right\| \left\| w_jf_j\right| L_p(\mathbb{R}^n)\right\| \\ &\le c'\mathrm{C}_w M_\kappa \left\| w_j\mathcal{F}^{-1}\varphi_j\mathcal{F}f\right| L_p(\mathbb{R}^n)\right\| \end{aligned} \tag{2.25}$$

We point out that the constant c in (2.25) is not depending on $j \in \mathbb{N}_0$ due to the homogenization in Remark 2.12. Furthermore, because of the definition of the weight sequence (2.3) also the constant C_w is independent of $j \in \mathbb{N}$ (see also the notation (2.16)).
The case $j = 0$ follows the same reasoning and we obtain (2.24) for all $j \in \mathbb{N}_0$. Finally, multiplying (2.24) by 2^{js} and taking the ℓ_q norm with respect to $j \in \mathbb{N}_0$ we get the theorem. □

Now, we give a simple multiplier criterion of Michlin-Hörmader type which is easy to verify.

Corollary 2.17: *Let $m(x)$ be a complex-valued infinitely differentiable function of at most polynomial growth on $\mathbb{R}^n$. If $N \in \mathbb{N}_0$, then we put*

$$\|m\|_N = \sup_{|\beta|\le N}\sup_{x\in\mathbb{R}^n}(1+|x|^2)^{\frac{|\beta|}{2}}\left|D^\beta m(x)\right| \ . \tag{2.26}$$

Further, let $\boldsymbol{w} \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$, $0 < p, q \le \infty$, $s \in \mathbb{R}$, then for $N \in \mathbb{N}$ with

$$N > n\left(\frac{1}{\min(1,p)} - \frac{1}{2}\right) + \alpha \ , \tag{2.27}$$

$$\tag{2.28}$$

there exists a constant $c > 0$ such that

$$\left\| \mathcal{F}^{-1}m\mathcal{F}f\right| B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\right\| \le c\|m\|_N \left\| f\right| B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\right\|$$

holds for all $f \in B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})$.

Proof: With the definition of M_N from (2.23) it is easy to check that $M_N \le c\|m\|_N$, where $c > 0$ is independent of $m(x)$. Hence, this corollary is a direct consequence of Theorem 2.16. □

Remark 2.18: This corollary is a direct generalization of Lemma 3.1 in [Mey97]. Meyer proved a condition as (2.26) of multipliers for the usual 2-microlocal case $w_j(x) = (1 + 2^j|x-x_0|)^{s'}$ with $x_0 \in \mathbb{R}^n$, $s' \in \mathbb{R}$ and $p = q = \infty$.

2.4 Lift property and equivalent norms

We introduce the lift operator as in the classical case of Besov spaces, [Tri83]. If $\sigma \in \mathbb{R}$, the operator I_σ is defined by

$$I_\sigma : f \mapsto \left(\langle\xi\rangle^\sigma \hat{f}\right)^\vee \tag{2.29}$$

where $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$.

Theorem 2.19: *Let $s,\sigma \in \mathbb{R}$ and $\boldsymbol{w} = \{w_j\}_{j\in\mathbb{N}_0} \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$. Moreover, let $0 < p \leq \infty$ and $0 < q \leq \infty$. Then I_σ maps $B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})$ isomorphically onto $B^{s-\sigma,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})$ and $\left\| I_\sigma f \right| B^{s-\sigma,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\|$ is an equivalent quasi-norm on $B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})$.*

Proof: To prove the theorem we show that

$$\begin{aligned}\left\| I_\sigma f \right| B^{s-\sigma,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\| &= \left(\sum_{j=0}^\infty 2^{j(s-\sigma)q} \left\| w_j(\varphi_j\langle\xi\rangle^\sigma\hat{f})^\vee \right| L_p(\mathbb{R}^n)\right\|^q\right)^{1/q} \\ &\leq c \left\| f \right| B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\| \ .\end{aligned}$$

The other direction follows by taking $I_{-\sigma}$. We take the special functions $\{\phi_j\}_{j\in\mathbb{N}_0} \in \mathcal{S}(\mathbb{R}^n)$ which satisfy (2.20)-(2.22). Then we have for $j \geq 1$

$$\left(\varphi_j\langle\xi\rangle^\sigma\hat{f}\right)^\vee = \left(\langle\xi\rangle^\sigma\phi_j(\xi)\varphi_j\hat{f}\right)^\vee \quad ,$$

and we define

$$M_j(\xi) := 2^{-\sigma j}\langle\xi\rangle^\sigma\phi_j(\xi) \ , \quad \text{whereas} \ , \quad \operatorname{supp}\varphi_j\hat{f} \subseteq \{\xi\in\mathbb{R}^n : |\xi| \leq 2^{j+1}\} \ .$$

Now, we can reason similar to the proof of Theorem 2.16 and apply (2.18) with $b = 2^{j+2}$ and $\kappa \in \mathbb{N}$ with $\kappa > n\left(\frac{1}{\min(1,p)} - \frac{1}{2}\right) + \alpha$ and obtain

$$\left\| w_j\left(2^{-\sigma j}\langle\xi\rangle^\sigma\phi(2^{-j}\xi)\varphi_j\hat{f}\right)^\vee \right| L_p(\mathbb{R}^n)\right\| \leq c \sup_{l\in\mathbb{N}_0} \left\| M_l(2^{l+2}\cdot) \right| W^\kappa_2(\mathbb{R}^n)\| \left\| w_j(\varphi_j\hat{f})^\vee \right| L_p(\mathbb{R}^n)\right\| \tag{2.30}$$

for all $j \in \mathbb{N}_0$ and $0 < p \leq \infty$. It remains to show that the Sobolev space norms are bounded. If $\beta \in \mathbb{N}^n_0$ is a multi-index with $|\beta| \leq \kappa$, we have

$$\begin{aligned}\left|D^\beta\left(M_l(2^{l+2}\cdot)\right)(x)\right| &= \left|D^\beta\left(2^{-\sigma l}\langle 2^{l+2}\cdot\rangle^\sigma\phi(4\cdot)\right)(x)\right| \\ &\leq 2^{2\sigma}\sum_{\gamma\leq\beta} c_{\beta,\gamma}\left|D^\gamma\left(2^{-2(l+2)}+|x|^2\right)^{\sigma/2}\right| \left|\left(D^{\beta-\gamma}\phi\right)(4x)\right| 4^{|\beta-\gamma|} \\ &\leq 2^{2(\sigma+\kappa)}\sup_{|\delta|\leq\kappa}\sup_{y\in\mathbb{R}^n}\left|\left(D^\delta\phi\right)(y)\right| \sum_{\gamma\leq\beta} c_{\beta,\gamma}\left|D^\gamma\left(2^{-2(l+2)}+|x|^2\right)^{\sigma/2}\right| \ .\end{aligned} \tag{2.31}$$

Since $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\operatorname{supp} \phi \subseteq \{\xi \in \mathbb{R}^n : \frac{1}{4} \le |\xi| \le 4\}$ we obtain

$$\sup_{|\delta| \le \kappa} \sup_{y \in \mathbb{R}^n} \left| \left(D^\delta \phi\right)(y) \right| \le c \; . \tag{2.32}$$

Furthermore, we have

$$\left| D^\gamma \left(2^{-2(l+2)} + |x|^2\right)^{\sigma/2} \right| \le c_{\sigma,\gamma} \left(2^{-2(l+2)} + |x|^2\right)^{\sigma/2-|\gamma|/2} \qquad \text{and} \tag{2.33}$$

$$\operatorname{supp} M_l(2^{l+2}\cdot) \subseteq \left\{ x \in \mathbb{R}^n : \frac{1}{16} \le |x| \le 1 \right\} \; . \tag{2.34}$$

Finally, we get from (2.31)-(2.34) for $0 < \sigma < \kappa$

$$\begin{aligned}
&\left| D^\beta \left(M_l(2^{l+2}\cdot)\right)(x) \right| \le c \sum_{\gamma \le \beta} \binom{\beta}{\gamma} c_{\sigma,\gamma} \left(2^{-2(l+2)} + |x|^2\right)^{\sigma/2-|\gamma|/2} \\
&\le c \sum_{\gamma \le \sigma} \binom{\beta}{\gamma} c_{\sigma,\gamma} \left(2^{-2(l+2)} + 1\right)^{\sigma/2-|\gamma|/2} + \sum_{\sigma < \gamma \le \kappa} \binom{\beta}{\gamma} c_{\sigma,\gamma} \left(2^{-2(l+2)} + \frac{1}{16}\right)^{\sigma/2-|\gamma|/2} \\
&\le c' \; .
\end{aligned}$$

This implies for all $l \in \mathbb{N}_0$ together with $\operatorname{supp} M_l(2^{l+2}\cdot) \subset [\frac{1}{16}, 1]$ that

$$\left\| M_l(2^{l+2}\cdot) \,\middle|\, W_2^\kappa(\mathbb{R}^n) \right\| = \left(\sum_{|\beta| \le \kappa} \left\| D^\beta \left(M_l(2^{l+2}\cdot)\right) \,\middle|\, L_2(\mathbb{R}^n) \right\|^2 \right)^{1/2} < \infty \; .$$

By a similar calculation as above for $j = 0$ we can show that for all $j \in \mathbb{N}_0$

$$\left\| w_j \left(2^{-\sigma j} \langle \xi \rangle^\sigma \varphi_j \hat{f}\right)^\vee \,\middle|\, L_p(\mathbb{R}^n) \right\| \le c \left\| w_j (\varphi_j \hat{f})^\vee \,\middle|\, L_p(\mathbb{R}^n) \right\| \; , \tag{2.35}$$

where c is independent of $j \in \mathbb{N}_0$.
Now, taking the ℓ_q quasi-norm in (2.30) leads to

$$\left\| I_\sigma f \,\middle|\, B_{pq}^{s-\sigma,mloc}(\mathbb{R}^n, \boldsymbol{w}) \right\| \le c \left\| f \,\middle|\, B_{pq}^{s,mloc}(\mathbb{R}^n, \boldsymbol{w}) \right\| \; .$$

This proves the theorem. □

The next theorem is a characterization of $B_{pq}^{s,mloc}(\mathbb{R}^n, \boldsymbol{w})$ given by equivalent norms related to the lift property and using derivatives. We follow closely Theorem 2.3.8 in [Tri83] and use Corollary 2.17 on multipliers and the lift theorem.

Theorem 2.20: *Let $s \in \mathbb{R}$, $\boldsymbol{w} = \{w_j\}_{j \in \mathbb{N}_0} \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$, $0 < p, q \le \infty$ and let $m \in \mathbb{N}_0$. Then*

$$\sum_{|\beta| \le m} \left\| D^\beta f \,\middle|\, B_{pq}^{s-m,mloc}(\mathbb{R}^n, \boldsymbol{w}) \right\| \;\; \textit{and} \;\; \left\| f \,\middle|\, B_{pq}^{s-m,mloc}(\mathbb{R}^n, \boldsymbol{w}) \right\| + \sum_{i=1}^n \left\| \frac{\partial^m f}{\partial x_i^m} \,\middle|\, B_{pq}^{s-m,mloc}(\mathbb{R}^n, \boldsymbol{w}) \right\|$$

are equivalent quasi-norms on $B_{pq}^{s,mloc}(\mathbb{R}^n, \boldsymbol{w})$.

Proof: First Step: The properties of the Fourier transform give us $\mathcal{F}D^\beta f = cx^\beta \mathcal{F}f$ where $x^\beta = x_1^{\beta_1}\cdots x_n^{\beta_n}$. We set

$$m(x) = x^\beta(1+|x|^2)^{-m/2} \tag{2.36}$$

and it is easy to check that $\|m\|_N \le c_N$ for $|\beta| \le m$. Consequently, we obtain by Corollary 2.17 with $N \in \mathbb{N}$ large enough and Theorem 2.19

$$\begin{aligned}\left\| D^\beta f \right| B^{s-m,mloc}_{p,q}(\mathbb{R}^n,\boldsymbol{w})\big\| &= c\left\| \mathcal{F}^{-1}x^\beta\mathcal{F}f \right| B^{s-m,mloc}_{p,q}(\mathbb{R}^n,\boldsymbol{w})\big\| \\ &= c\left\| \mathcal{F}^{-1}x^\beta(1+|x|^2)^{-m/2}\mathcal{F}\mathcal{F}^{-1}(1+|x|^2)^{m/2}\mathcal{F}f \right| B^{s-m,mloc}_{p,q}(\mathbb{R}^n,\boldsymbol{w})\big\| \\ &\le c'\|m\|_N \left\| I_m f \right| B^{s-m,mloc}_{p,q}(\mathbb{R}^n,\boldsymbol{w})\big\| \le c'' \left\| f \right| B^{s,mloc}_{p,q}(\mathbb{R}^n,\boldsymbol{w})\big\| \ .\end{aligned} \tag{2.37}$$

Second Step: Now, we assume that $f \in B^{s-m,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})$ and $\frac{\partial^m f}{\partial x_i^m} \in B^{s-m,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})$ for $i = 1,\ldots,n$. We want to show that f belongs to $B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})$. Theorem 2.19 gives us

$$\begin{aligned}\left\| f \right| B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\big\| &\le c\left\| I_m f \right| B^{s-m,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\big\| \\ &= c\left(\sum_{j=0}^\infty 2^{j(s-m)q}\left\| w_j\mathcal{F}^{-1}(1+|x|^2)^{m/2}\varphi_j\mathcal{F}f \right| L_p(\mathbb{R}^n)\big\|^q\right) \ .\end{aligned}$$

We define an infinitely differentiable function $\varrho(t)$, $t \in \mathbb{R}$, which is odd and fulfills

$$\begin{aligned}&\varrho(t) = 0 \quad \text{if} \quad 0 \le t \le 1/2 \\ &\varrho(t) = 1 \quad \text{if} \quad t \ge 1 \ .\end{aligned}$$

Then by Corollary 2.17

$$m(x) := (1+|x|^2)^{m/2}\left[1+\sum_{i=1}^n(\varrho(x_i)x_i)^m\right]^{-1} \tag{2.38}$$

is a multiplier for $B^{s-m,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})$. Hence, we obtain in using Corollary 2.17

$$\begin{aligned}\left\| f \right| B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\big\| &\le c\left\| \mathcal{F}^{-1}(1+|x|^2)^{m/2}\mathcal{F}f \right| B^{s-m,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\big\| \\ &\le c\left\| \mathcal{F}^{-1}m(x)\mathcal{F}\mathcal{F}^{-1}\left[1+\sum_{i=1}^n(\varrho(x_i)x_i)^m\right]\mathcal{F}f \right| B^{s-m,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\Bigg\| \\ &\le c'\left\| f \right| B^{s-m,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\big\| + c'\sum_{j=1}^n\left\| \mathcal{F}^{-1}(\varrho(x_j)x_j)^m\mathcal{F}f \right| B^{s-m,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\big\| \ .\end{aligned}$$

Finally, we have $x_j\mathcal{F}f = c\mathcal{F}\frac{\partial^m f}{\partial x_i^m}$. Consequently, using that ϱ is a one-dimensional multiplier we obtain

$$\left\| f \right| B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\big\| \le c\left\| f \right| B^{s-m,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\big\| + c\sum_{i=1}^n\left\| \frac{\partial^m f}{\partial x_i^m} \right| B^{s-m,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\Bigg\| \ .$$

This and (2.37) prove the theorem.

□

Now, we present a characterization of the 2-microlocal spaces with the special weight sequence $w_j(x) = (1 + 2^j \operatorname{dist}(x, U))^{s'}$ for $U \subseteq \mathbb{R}^n$.

Definition 2.21: *Let $\{\varphi_j\}_{j\in\mathbb{N}_0} \in \Phi(\mathbb{R}^n)$ be a resolution of unity. Let $U \subseteq \mathbb{R}^n$ be bounded and $s' \in \mathbb{R}$ be fixed. Further, let $0 < p \le \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Then we define*

$$B^{s,s'}_{pq}(\mathbb{R}^n, U) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \left\| f |\, B^{s,s'}_{pq}(\mathbb{R}^n, U) \right\| < \infty \right\} \quad , \textit{where}$$

$$\left\| f |\, B^{s,s'}_{pq}(\mathbb{R}^n, U) \right\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| (1 + 2^j \operatorname{dist}(x, U))^{s'} (\varphi_j \hat{f})^{\vee} \right| L_p(\mathbb{R}^n) \right\|^q \right)^{1/q} ,$$

with the usual modifications if p or q are equal to infinity.

Remark 2.22: In slight abuse of notation we write $B^{s,s'}_{pq}(\mathbb{R}^n, x_0)$ if $U = \{x_0\} \subset \mathbb{R}^n$. If $U = \{x_0\} \subset \mathbb{R}^n$ then $B^{s,s'}_{\infty\infty}(\mathbb{R}^n, x_0) = C^{s,s'}_{x_0}(\mathbb{R}^n)$, see [JaMey96, Definition 1.1]. For $p = q = 2$ we get $B^{s,s'}_{22}(\mathbb{R}^n, x_0) = H^{s,s'}_{x_0}(\mathbb{R}^n)$. Both types are the 2-microlocal spaces introduced by Bony [Bo84] and Jaffard [Ja91].

Corollary 2.23: *Let $s, s' \in \mathbb{R}$ and let $U \subseteq \mathbb{R}^n$. Further, let $0 < p, q \le \infty$ and $m \in \mathbb{N}_0$, then the following statements are equivalent*

(i) $f \in B^{s,s'}_{pq}(\mathbb{R}^n, U)$

(ii) $D^\beta f \in B^{s-m,s'}_{pq}(\mathbb{R}^n, U)$ *for all* $0 \le |\beta| \le m$

(iii) $f \in B^{s-m,s'}_{pq}(\mathbb{R}^n, U)$ *and* $\frac{\partial^m f}{\partial x_i^m} \in B^{s-m,s'}_{pq}(\mathbb{R}^n, U)$ *for each* $i = 1, \ldots, n$.

Remark 2.24: This corollary coincides essentially with Corollary 3.1 in [Mey97] for the special case $p = q = \infty$ and $U = \{x_0\} \subset \mathbb{R}^n$.

2.5 Embedding theorems

2.5.1 General embeddings

For the spaces $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ introduced above we want to show some general embedding theorems. We follow closely [Tri83], see Proposition 2.3.2/2 and Theorem 2.7.1. We say a quasi-Banach space A_1 is continuously embedded in another quasi-Banach space A_2, $A_1 \hookrightarrow A_2$, if $A_1 \subseteq A_2$ and there is a $c > 0$ such that $\|a|\, A_2\| \le c \|a|\, A_1\|$ for all $a \in A_1$. First, we present an embedding theorem which connects the 2-microlocal Besov spaces with the usual weighted Besov spaces [EdTri96]. We recall the spaces $B^s_{p,q}(\mathbb{R}^n, \alpha)$ which are weighted Besov spaces, with respect to the weight $\langle x\rangle^\alpha = (1 + |x|^2)^{\alpha/2}$ for $\alpha \in \mathbb{R}$.

Theorem 2.25: *Let $\boldsymbol{w} \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$, $s \in \mathbb{R}$ and $0 < p, q \le \infty$, then*

$$B^{s+\alpha_2}_{pq}(\mathbb{R}^n, \alpha) \hookrightarrow B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \hookrightarrow B^{s-\alpha_1}_{pq}(\mathbb{R}^n, -\alpha) .$$

Proof: Using the properties (2.3) and (2.4) we obtain

$$w_j(x) \le 2^{j\alpha_2} w_0(x) \le \mathrm{C} 2^{j\alpha_2} w_0(0)(1 + |x|^2)^{\alpha/2}$$

$$w_j(x) \ge 2^{-j\alpha_1} w_0(x) \ge \frac{1}{\mathrm{C}} 2^{-j\alpha_1} w_0(0)(1 + |x|^2)^{-\alpha/2}$$

for all $x \in \mathbb{R}^n$ and every $j \in \mathbb{N}_0$. It follows immediately

$$c_1 2^{-j\alpha_1} \left\| (1+|x|^2)^{-\alpha/2} (\varphi_j \hat{f})^\vee \middle| L_p(\mathbb{R}^n) \right\| \le \left\| w_j (\varphi_j \hat{f})^\vee \middle| L_p(\mathbb{R}^n) \right\|$$
$$\le c_2 2^{j\alpha_2} \left\| (1+|x|^2)^{\alpha/2} (\varphi_j \hat{f})^\vee \middle| L_p(\mathbb{R}^n) \right\|$$

and therefrom the theorem. □

The following is a consequence of the above theorem and $B^s_{pq}(\mathbb{R}^n) \hookrightarrow L_{\max(1,p)}(\mathbb{R}^n)$ for $s > \sigma_p$ and the observation that $\| f | B^\sigma_{pq}(\mathbb{R}^n, \alpha) \| \sim \| \langle x \rangle^\alpha f | B^\sigma_{pq}(\mathbb{R}^n) \|$ (see [EdTri96, Theorem 4.2.2]). As usual, the number σ_p is defined by

$$\sigma_p = n\left(\frac{1}{p} - 1\right)_+ = \begin{cases} n\left(\frac{1}{p} - 1\right) & , \text{ for } 0 < p < 1 \\ 0 & , \text{ for } 1 \le p \le \infty. \end{cases} \tag{2.39}$$

Corollary 2.26: *Let and $\boldsymbol{w} \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$ and let $0 < p, q \le \infty$, then for $s > \sigma_p + \alpha_1$*

$$B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \hookrightarrow L_{\max(1,p)}(\mathbb{R}^n, \langle x \rangle^{-\alpha}) \ .$$

We need a special weighted version of Nikol'skij's inequality. We recall Proposition 1.4.3 in [SchmTri87], adapted to the admissible weights we consider.

Proposition 2.27: *Let ϱ be an admissible weight satisfying (2.5) and let $0 < p \le q \le \infty$. Let Ω be a compact subset of $\mathbb{R}^n$ and $\beta \in \mathbb{N}^n_0$, then there exists a positive constant c such that*

$$\| \varrho D^\beta \varphi | L_q(\mathbb{R}^n) \| \le c \| \varrho \varphi | L_p(\mathbb{R}^n) \|$$

holds for all $\varphi \in L_p(\mathbb{R}^n, \varrho)$ with $\operatorname{supp} \hat{\varphi} \subseteq \Omega$.

Remark 2.28: It is not hard to formulate a homogenized version of this proposition. We assume that $\operatorname{supp} \mathcal{F}\varphi \subset B_b(0)$. Then $\operatorname{supp} \mathcal{F}\varphi(b^{-1}\cdot) \subset B_1(0)$. Furthermore, we assume that our weight function ϱ satisfies

$$0 < \varrho(x) \le \mathrm{C}_\varrho \varrho(y)(1 + b|x-y|)^{\alpha_\varrho} \quad \text{for all } x, y \in \mathbb{R}^n \ .$$

Then we can use Proposition 2.27 with $\varrho(b^{-1}x)$ and $\varphi(b^{-1}x)$ instead of ϱ and φ and derive

$$\| \varrho D^\beta \varphi | L_q(\mathbb{R}^n) \| \le c 2^{|\beta| + n\left(\frac{1}{p} - \frac{1}{q}\right)} \| \varrho \varphi | L_p(\mathbb{R}^n) \| \ . \tag{2.40}$$

The constant c does depend on C_ϱ but it is independent on b and on the special choice of the weight ϱ (see Remark 2 [SchmTri87, 1.4.2]).

Theorem 2.29: *Let $\boldsymbol{w} \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$ and $0 < p \le q \le \infty$. Let $f_j \in L_p(\mathbb{R}^n, w_j)$ with $\operatorname{supp} \mathcal{F} f_j \subset B_{d2^j}$ for fixed $d > 0$. Then for $\beta \in \mathbb{N}_0$ there exists a constant $c > 0$ such that*

$$\| w_j f_j | L_q(\mathbb{R}^n) \| \le c 2^{|\beta| + n\left(\frac{1}{p} - \frac{1}{q}\right)} \| w_j f_j | L_p(\mathbb{R}^n) \|$$

holds, where c is independent on $j \in \mathbb{N}_0$.

Proof: The theorem is an easy application of the homogenization procedure above. The condition (2.3) can be rewritten in

$$w_j(x) \le \mathrm{C}' w_j(y)(1 + d2^j|x-y|)^\alpha \ ,$$

where $\mathrm{C}' = \mathrm{C}\max(1, d^{-\alpha})$ is a new constant which is independent on $j \in \mathbb{N}_0$. Now, applying (2.40) with $b = d2^j$ proves the theorem. $\square$

Theorem 2.30: *Let $s \in \mathbb{R}$ and $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ and $\boldsymbol{\varrho} \in \mathcal{W}^{\beta}_{\beta_1,\beta_2}$.*

(i) If $0 < p \le \infty$, $0 < q_1 \le q_2 \le \infty$ and $\frac{w_j(x)}{\varrho_j(x)} \le c$ for all $j \in \mathbb{N}_0$ and $x \in \mathbb{R}^n$, then

$$B^{s,mloc}_{pq_1}(\mathbb{R}^n, \boldsymbol{\varrho}) \hookrightarrow B^{s,mloc}_{pq_2}(\mathbb{R}^n, \boldsymbol{w}) \ . \tag{2.41}$$

(ii) If $0 < p \le \infty$, $0 < q_1 \le \infty$, $0 < q_2 \le \infty$ and $\frac{w_j(x)}{\varrho_j(x)} \le c$ for all $j \in \mathbb{N}_0$ and $x \in \mathbb{R}^n$, then for all $\varepsilon > 0$

$$B^{s,mloc}_{pq_1}(\mathbb{R}^n, \boldsymbol{\varrho}) \hookrightarrow B^{s-\varepsilon,mloc}_{pq_2}(\mathbb{R}^n, \boldsymbol{w}) \ . \tag{2.42}$$

(iii) If $0 < p_1 \le p_2 \le \infty$, $0 < q \le \infty$ and $s_1, s_2 \in \mathbb{R}$ with

$$\frac{w_j(x)}{\varrho_j(x)} \le c 2^{j\left(s_1 - s_2 - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)\right)} \tag{2.43}$$

for all $j \in \mathbb{N}_0$ and $x \in \mathbb{R}^n$, then we have

$$B^{s_1,mloc}_{p_1 q}(\mathbb{R}^n, \boldsymbol{\varrho}) \hookrightarrow B^{s_2,mloc}_{p_2 q}(\mathbb{R}^n, \boldsymbol{w}) \ . \tag{2.44}$$

Proof: The proof of (2.41) follows from the embedding $\ell_{q_1} \hookrightarrow \ell_{q_2}$ for $q_1 \le q_2$. To get (2.42) we estimate

$$\left(\sum_{j=0}^{\infty} 2^{j(s-\varepsilon)q_1} |b_k|^{q_1}\right)^{1/q_1} \le \sup_{j\in\mathbb{N}_0} 2^{js}|b_j| \left(\sum_{k=0}^{\infty} 2^{-k\varepsilon q_1}\right)^{1/q_1} \le c \sup_{j\in\mathbb{N}_0} 2^{js}|b_j| \ ,$$

which gives us $B^{s,mloc}_{p\infty}(\mathbb{R}^n, \boldsymbol{w}) \hookrightarrow B^{s-\varepsilon,mloc}_{pq_1}(\mathbb{R}^n, \boldsymbol{w})$ and the rest follows by (i).
To prove (2.44) we use Theorem 2.29 and obtain

$$\left\| w_j(\varphi_j \hat{f})^\vee \middle| L_{p_2}(\mathbb{R}^n)\right\| \le c 2^{jn\left(\frac{1}{p_1} - \frac{1}{p_2}\right)} \left\| w_j(\varphi_j \hat{f})^\vee \middle| L_{p_1}(\mathbb{R}^n)\right\| \ ,$$

for all $j \in \mathbb{N}_0$, where the constant c is independent of $j \in \mathbb{N}_0$. After multiplying this inequality by 2^{js_2} and using condition (2.43) on the weight sequences, we get

$$2^{js_2} \left\| w_j(\varphi_j \hat{f})^\vee \middle| L_{p_2}(\mathbb{R}^n)\right\| \le c' 2^{js_1} \left\| \varrho_j(\varphi_j \hat{f})^\vee \middle| L_{p_1}(\mathbb{R}^n)\right\| \ .$$

Finally we apply the ℓ_q quasi-norm to find the desired result. $\square$

Remark 2.31: (a) The question arises, if it is possible to find a weight sequence $\boldsymbol{\varrho} \in \mathcal{W}^{\beta}_{\beta_1,\beta_2}$ which fulfills $\frac{w_j(x)}{\varrho_j(x)} \leq c2^{j\mu}$ for a given weight sequence $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ and $\mu \in \mathbb{R}$.

This can be solved in defining

$$\varrho_j(x) = 2^{-j\mu} w_j(x) \quad \text{for all } j \in \mathbb{N}_0 \text{ and } x \in \mathbb{R}^n.$$

This weight sequence obviously ensures the demanded condition but $\alpha_1,\ \alpha_2 \geq 0$ have to be large enough such that $\beta_1 = \alpha_1 + \mu$ and $\beta_2 = \alpha_2 - \mu$ are still positive.

(b) If $\boldsymbol{w} = \boldsymbol{\varrho}$ in the above theorem, then condition (2.43) becomes

$$s_2 - \frac{n}{p_2} \leq s_1 - \frac{n}{p_1}$$

which is the usual condition for this embedding.

With minor modifications we have an analogous theorem to Theorem 2.3.3 in [Tri83]. The proof is essentially the same. One only has to bring in the weight sequence and use its properties (2.3) and (2.4). Also the weighted Nikol'skij inequality (Proposition 2.27) and Section 1.5 in [SchmTri87] has to be used in the proof as a replacement for the unweighted ones in the proof in [Tri83, 2.3.3].

Theorem 2.32: *Let $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $s \in \mathbb{R}$ and $0 < p, q \leq \infty$, then*

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \hookrightarrow \mathcal{S}'(\mathbb{R}^n) \quad \textit{holds.} \tag{2.45}$$

If $s \in \mathbb{R}$ and $0 < p, q < \infty$, then $\mathcal{S}(\mathbb{R}^n)$ is dense in $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$.

2.5.2 Embeddings for 2-microlocal Besov spaces

In this subsection we present some special embedding theorems for the weight sequence of 2-microlocal weights, $w_j(x) = (1+2^j \operatorname{dist}(x, U))^{s'}$ for fixed and bounded $U \subseteq \mathbb{R}^n$ and $s' \in \mathbb{R}$. The spaces $B^{s,s'}_{pq}(\mathbb{R}^n, U)$ have been defined in Definition 2.21. As shown in Example 2.5, the weight sequence belongs to $\mathcal{W}^{|s'|}_{\max(0,-s'),\max(0,s')}$. We recall the spaces $B^s_{pq}(\mathbb{R}^n, \alpha)$, with respect to the weight $\langle x \rangle^{\alpha} = (1 + |x|^2)^{\alpha/2}$ for $\alpha \in \mathbb{R}$. An easy consequence of Theorem 2.25 and Theorem 2.30 is the following.

Theorem 2.33: *Let $s \in \mathbb{R}$, $U \subseteq V \subset \mathbb{R}^n$ are bounded and let $0 < p, q \leq \infty$.*

(i) For $s' \in \mathbb{R}$ and $U = \{x_0\} \in \mathbb{R}^n$ we have

$$B^{s,s'}_{pq}(\mathbb{R}^n, x_0) \hookrightarrow C^{s-\frac{n}{p},s'}_{x_0}(\mathbb{R}^n)\ .$$

(ii) For $s' \geq 0$ we have

$$B^{s+s'}_{pq}(\mathbb{R}^n, s') \hookrightarrow B^{s,s'}_{pq}(\mathbb{R}^n, U) \hookrightarrow B^{s,s'}_{pq}(\mathbb{R}^n, V) \hookrightarrow B^{s}_{pq}(\mathbb{R}^n, -s')\ .$$

(iii) For $s' \geq 0$ we have

$$B^{s,s'}_{pq}(\mathbb{R}^n, U) \hookrightarrow B^{s,s'}_{pq}(\mathbb{R}^n, V) \hookrightarrow B^{s}_{pq}(\mathbb{R}^n) \hookrightarrow B^{s,-s'}_{pq}(\mathbb{R}^n, V) \hookrightarrow B^{s,-s'}_{pq}(\mathbb{R}^n, U) \ .$$

(iv) For $s' \geq t'$ and $s \geq t$ we have

$$B^{s,s'}_{pq}(\mathbb{R}^n, U) \hookrightarrow B^{t,t'}_{pq}(\mathbb{R}^n, U) \ .$$

Remark 2.34: Now one can ask, if it is possible in Theorem 2.33 (*iv*) that the weight sequences are able to affect the condition on s and t in the same manner as in (2.43). The answer is negative, since we have

$$\frac{(1+2^j \operatorname{dist}(U,x))^{t'}}{(1+2^j \operatorname{dist}(U,x))^{s'}} = (1+2^j \operatorname{dist}(U,x))^{t'-s'} \leq c 2^{j(s-t)}$$

if, and only if,

$$s \geq t \quad \text{and} \quad s' \geq t' \ .$$

To prove the reverse statement, we differ two cases.
<u>First case:</u> We suppose $s < t$ and that we have a $c > 0$ with

$$(1+2^j \operatorname{dist}(U,x))^{t'-s'} \leq c 2^{-j\varepsilon} \quad \text{for an } \varepsilon > 0, \text{ all } j \in \mathbb{N}_0 \text{ and all } x \in \mathbb{R}^n. \tag{2.46}$$

We choose $x^j \in \mathbb{R}^n$ with $\operatorname{dist}(x^j, U) = 2^{-j}$ (U is bounded) and get

$$(1+2^j \operatorname{dist}(U,x^j))^{t'-s'} = 2^{t'-s'}$$

which is constant for all possible $s', t' \in \mathbb{R}$ and does not depend on $j \in \mathbb{N}_0$. This is a contradiction to (2.46) since $2^{-j\varepsilon} \to 0$ for $j \to \infty$.
<u>Second case:</u> We suppose $s' < t'$ and that we can find a $c > 0$ with

$$(1+2^j \operatorname{dist}(U,x))^{\varepsilon} \leq c 2^{j(s-t)} \quad \text{for an } \varepsilon > 0, \text{ all } j \in \mathbb{N}_0 \text{ and all } x \in \mathbb{R}^n. \tag{2.47}$$

Now, the right hand side is not depending on $x \in \mathbb{R}^n$ and we can easily find a sequence $\{x^k\} \subset \mathbb{R}^n$, for fixed $j \in \mathbb{N}_0$ such that the left hand side goes to infinity. This contradicts (2.47) for all possible $s, t \in \mathbb{R}$.

Corollary 2.35: *Let $s \geq s' \geq 0$ and let $0 < p, q \leq \infty$. Further, if $U \subseteq \mathbb{R}^n$, then*

$$B^{s,s'}_{pq}(\mathbb{R}^n, U) \hookrightarrow B^{s}_{pq}(\mathbb{R}^n) \hookrightarrow B^{s,-s}_{pq}(\mathbb{R}^n, U) \ .$$

Remark 2.36: Corollary 2.35 coincides partially with Proposition 1.3 (1) and (2) in [JaMey96] for $p = q = \infty$ and $U = \{x_0\}$ and with Theorem 3.2 in [MoYa04] with $p = q \geq 1$ and U be an open subset or $U = \{x_0\} \subset \mathbb{R}^n$.
In the mentioned papers local versions of $B^{s,s'}_{pq}(\mathbb{R}^n, U)$ have been used to treat further kinds of embeddings in the scale of $B^{s,s'}_{pq}(\mathbb{R}^n, U)$ (see also Section 7.1).

3 Local Means

3.1 Preliminaries

In this part we present the main technical tool. We characterize the spaces $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ by so called *local means*. We follow closely the method presented by Rychkov [Ry99] and by Vybiral [Vyb06].
Recall the specific system $\{\varphi_j\}_{j\in\mathbb{N}_0} \in \Phi(\mathbb{R}^n)$ which we fix now for the rest of our work: Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ with

$$\varphi_0(x) = \begin{cases} 1 \ , & \text{if } |x| \leq 1 \\ 0 \ , & \text{if } |x| \geq 2 \end{cases} \quad . \tag{3.1}$$

We put $\varphi(x) = \varphi_0(x) - \varphi_0(2x)$ and

$$\varphi_j(x) = \varphi(2^{-j}x) \qquad \text{for all } x \in \mathbb{R}^n \text{ and all } j \in \mathbb{N}.$$

3.1.1 The Peetre maximal operator

The Peetre maximal operator was introduced by Jaak Peetre in [Pe75]. The operator assigns to each system $\{\psi_j\}_{j\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^n)$, to each distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ and to each number $a > 0$ the following quantities

$$\sup_{y\in\mathbb{R}^n} \frac{|(\psi_k \hat{f})^\vee(y)|}{1 + |2^k(y-x)|^a} \ , \quad x \in \mathbb{R}^n \ ,k \in \mathbb{N}_0 \ . \tag{3.2}$$

Since $\psi_k \in \mathcal{S}(\mathbb{R}^n)$ for all $k \in \mathbb{N}_0$ the operator is well-defined because $(\psi_k \hat{f})^\vee = c(\psi_k^\vee * f)$ is well-defined for every distribution $f \in \mathcal{S}'(\mathbb{R}^n)$.
Given a system $\{\psi_k\}_{k\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^n)$, we set $\Psi_k = \hat{\psi}_k \in \mathcal{S}(\mathbb{R}^n)$ and reformulate the Peetre maximal operator (3.2) for every $f \in \mathcal{S}'(\mathbb{R}^n)$ and $a > 0$ as

$$(\Psi_k^* f)_a(x) = \sup_{y\in\mathbb{R}^n} \frac{|(\Psi_k * f)(y)|}{1 + |2^k(y-x)|^a} \ , \quad x \in \mathbb{R}^n \text{ and } k \in \mathbb{N}_0 \ . \tag{3.3}$$

3.1.2 Helpful lemmas

Before proving the local means characterization we recall some technical lemmas without proof, which appeared in the papers of Rychkov [Ry99] and Vybiral [Vyb06]. The first lemma describes the use of the so called moment conditions.
Lemma 3.1 ([Ry99], Lemma 1)**:** *Let $g, h \in \mathcal{S}(\mathbb{R}^n)$ and let $M \in \mathbb{N}_0$. Suppose that*

$$(D^\beta \hat{g})(0) = 0 \quad \textit{for} \quad 0 \leq |\beta| < M \ . \tag{3.4}$$

Then for each $N \in \mathbb{N}_0$ there is a constant C_N such that

$$\sup_{z\in\mathbb{R}^n} |(g_t * h)(z)|(1+|z|^N) \le C_N t^M \ , \quad \textit{for} \quad 0 < t < 1 \ , \tag{3.5}$$

where $g_t(x) = t^{-n} g(x/t)$.

Remark 3.2: If $M = 0$, then condition (3.4) is empty.

The next lemma is a discrete convolution inequality which we will need later on.

Lemma 3.3 ([Ry99], Lemma 2)**:** *Let $0 < p, q \le \infty$ and $\delta > 0$. Let $\{g_k\}_{k\in\mathbb{N}_0}$ be a sequence of non-negative measurable functions on $\mathbb{R}^n$ and let*

$$G_\nu(x) = \sum_{k=0}^{\infty} 2^{-|\nu-k|\delta} g_k(x) \ , \qquad x \in \mathbb{R}^n \ , \nu \in \mathbb{N}_0 \ . \tag{3.6}$$

Then there is some constant $c = c(p,q,\delta)$ such that

$$\| G_k | \, \ell_q(L_p) \| \le c \, \| g_k | \, \ell_q(L_p) \| \ . \tag{3.7}$$

Lemma 3.4 ([Ry99], Lemma 3)**:** *Let $0 < r \le 1$ and let $\{\gamma_\nu\}_{\nu\in\mathbb{N}_0}$, $\{\beta_\nu\}_{\nu\in\mathbb{N}_0}$ be two sequences taking values in $(0,\infty)$. Assume that for some $N^0 \in \mathbb{N}_0$,*

$$\gamma_\nu = O(2^{\nu N^0}) \ , \quad \textit{for } \nu \to \infty \ . \tag{3.8}$$

Furthermore, we assume that for any $N \in \mathbb{N}$

$$\gamma_\nu \le C_N \sum_{k=0}^{\infty} 2^{-kN} \beta_{k+\nu} \gamma_{k+\nu}^{1-r} \ , \quad \nu \in \mathbb{N}_0 \ , \quad C_N < \infty$$

holds, then for any $N \in \mathbb{N}$

$$\gamma_\nu^r \le C_N \sum_{k=0}^{\infty} 2^{-kN} \beta_{k+\nu} \ , \quad \nu \in \mathbb{N}_0 \tag{3.9}$$

holds with the same constants C_N.

3.1.3 Comparison of different Peetre maximal operators

In this subsection we present an inequality between different Peetre maximal operators. We start with two given functions $\psi_0, \psi_1 \in \mathcal{S}(\mathbb{R}^n)$. We define

$$\psi_j(x) = \psi_1(2^{-j+1}x) \ , \quad \text{for } x \in \mathbb{R}^n \text{ and } j \in \mathbb{N}. \tag{3.10}$$

Furthermore, for all $j \in \mathbb{N}_0$ we write $\Psi_j = \hat{\psi}_j$ and in an analogous manner we define Φ_j from two starting functions $\phi_0, \phi_1 \in \mathcal{S}(\mathbb{R}^n)$.
Using this notation we are ready to formulate the theorem.

Theorem 3.5: *Let* $\boldsymbol{w} = \{w_j\}_{j\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $0 < p, q \le \infty$ *and* $s, a \in \mathbb{R}$ *with* $a > 0$. *Moreover, let* $R \in \mathbb{N}_0$ *with* $R > s + \alpha_2$,

$$D^{\beta}\psi_1(0) = 0 \ , \quad 0 \le |\beta| < R \tag{3.11}$$

and for some $\varepsilon > 0$

$$|\phi_0(x)| > 0 \quad on \quad \{x \in \mathbb{R}^n : |x| < \varepsilon\} \tag{3.12}$$
$$|\phi_1(x)| > 0 \quad on \quad \{x \in \mathbb{R}^n : \varepsilon/2 < |x| < 2\varepsilon\} \tag{3.13}$$

then

$$\left\| 2^{ks}(\Psi_k^* f)_a w_k \right| \ell_q(L_p) \right\| \le c \left\| 2^{ks}(\Phi_k^* f)_a w_k \right| \ell_q(L_p) \right\| \tag{3.14}$$

holds for every $f \in \mathcal{S}'(\mathbb{R}^n)$.

Proof: We have the fixed resolution of unity from (3.1) and define the functions $\{\lambda_j\}_{j\in\mathbb{N}_0}$ by

$$\lambda_j(x) = \frac{\varphi_j\left(\frac{2x}{\varepsilon}\right)}{\phi_j(x)} \ .$$

It follows from the *Tauberian conditions* (3.12) and (3.13) that they satisfy

$$\sum_{j=0}^{\infty} \lambda_j(x)\phi_j(x) = 1 \ , \quad x \in \mathbb{R}^n \tag{3.15}$$
$$\lambda_j(x) = \lambda_1(2^{-j+1}x) \ , \quad x \in \mathbb{R}^n \ , \quad j \in \mathbb{N} \tag{3.16}$$
$$\operatorname{supp}\lambda_0 \subset \{x \in \mathbb{R}^n : |x| \le \varepsilon\} \quad \text{and} \quad \operatorname{supp}\lambda_1 \subset \{x \in \mathbb{R}^n : \varepsilon/2 \le |x| \le 2\varepsilon\} \ . \tag{3.17}$$

Furthermore, we denote $\Lambda_k = \hat{\lambda_k}$ for $k \in \mathbb{N}_0$ and obtain together with (3.15) the following identities (convergence in $\mathcal{S}'(\mathbb{R}^n)$)

$$f = \sum_{k=0}^{\infty} \Lambda_k * \Phi_k * f \ , \qquad \Psi_\nu * f = \sum_{k=0}^{\infty} \Psi_\nu * \Lambda_k * \Phi_k * f \ . \tag{3.18}$$

We have

$$\begin{aligned} |(\Psi_\nu * \Lambda_k * \Phi_k * f)(y)| &\le \int_{\mathbb{R}^n} |(\Psi_\nu * \Lambda_k)(z)||(\Phi_k * f)(y-z)|dz \\ &\le (\Phi_k^* f)_a(y) \int_{\mathbb{R}^n} |(\Psi_\nu * \Lambda_k)(z)|(1 + |2^k z|^a)dz \\ &=: (\Phi_k^* f)_a(y) I_{\nu,k} \ , \end{aligned} \tag{3.19}$$

where

$$I_{\nu,k} := \int_{\mathbb{R}^n} |(\Psi_\nu * \Lambda_k)(z)|(1 + |2^k z|^a)dz \ .$$

According to Lemma 3.1 we get

$$I_{\nu,k} \le c \begin{cases} 2^{(k-\nu)R} & ,k \le \nu \\ 2^{(\nu-k)(a+|s|+1+\alpha_1)} & ,\nu \le k \ . \end{cases} \tag{3.20}$$

Namely, we have for $1 \le k < \nu$ with the change of variables $2^k z \mapsto z$

$$\begin{aligned} I_{\nu,k} &= 2^{-n} \int\limits_{\mathbb{R}^n} |(\Psi_{\nu-k} * \Lambda_1(\cdot/2))(z)|(1+|z|^a)dz \\ &\le c \sup_{z\in\mathbb{R}^n} |(\Psi_{\nu-k} * \Lambda_1(\cdot/2))(z)|(1+|z|)^{a+n+1} \le c2^{(k-\nu)R} \ . \end{aligned}$$

Similarly, we get for $1 \le \nu < k$ with the substitution $2^\nu z \mapsto z$

$$\begin{aligned} I_{\nu,k} &= 2^{-n} \int\limits_{\mathbb{R}^n} |(\Psi_1(\cdot/2) * \Lambda_{k-\nu})(z)|(1+|2^{k-\nu}z|^a)dz \\ &\le c2^{(\nu-k)(M-a)} \ . \end{aligned}$$

M can be taken arbitrarily large because Λ_1 has infinite vanishing moments. Taking $M = 2a + |s| + \alpha_1 + 1$ we derive (3.20) for the cases $k, \nu \ge 1$ with $k \ne \nu$. The missing cases can be treated separately in an analogous manner. The needed moment conditions are always satisfied by (3.11) and (3.17). The case $k = \nu = 0$ is covered by the constant c in (3.20).
Furthermore, we have

$$\begin{aligned} (\Phi_k^* f)_a(y) &\le (\Phi_k^* f)_a(x)(1+|2^k(x-y)|^a) \\ &\le (\Phi_k^* f)_a(x)(1+|2^\nu(x-y)|^a)\max(1, 2^{(k-\nu)a}) \ . \end{aligned}$$

We put this into (3.19) and get

$$\sup_{y\in\mathbb{R}^n} \frac{|(\Psi_\nu * \Lambda_k * \Phi_k * f)(y)|}{1+|2^\nu(x-y)|^a} \le c(\Phi_k^* f)_a(x) \begin{cases} 2^{(k-\nu)R} & ,k \le \nu \\ 2^{(\nu-k)(|s|+1+\alpha_1)} & ,k \ge \nu \ . \end{cases}$$

Multiplying both sides with $w_\nu(x)$ and using

$$w_\nu(x) \le w_k(x) \begin{cases} 2^{(k-\nu)(-\alpha_2)} & ,k \le \nu \\ 2^{(\nu-k)(-\alpha_1)} & ,k \ge \nu \end{cases} , \tag{3.21}$$

leads us to

$$\sup_{y\in\mathbb{R}^n} \frac{|(\Psi_\nu * \Lambda_k * \Phi_k * f)(y)|}{1+|2^\nu(x-y)|^a} w_\nu(x) \le c(\Phi_k^* f)_a(x) w_k(x) \begin{cases} 2^{(k-\nu)(R-\alpha_2)} & ,k \le \nu \\ 2^{(\nu-k)(|s|+1)} & ,k \ge \nu \ . \end{cases}$$

This inequality together with (3.18) gives for $\delta := \min(1, R - \alpha_2 - s) > 0$

$$2^{\nu s}(\Psi_\nu^* f)_a(x) w_\nu(x) \le c \sum_{k=0}^{\infty} 2^{-|k-\nu|\delta} 2^{ks} (\Phi_k^* f)_a(x) w_k(x) \ , \quad x \in \mathbb{R}^n \ .$$

Then, Lemma 3.3 yields immediately the desired result.

□

Remark 3.6: The conditions (3.11) are usually called *moment conditions* while (3.12) and (3.13) are the so called *Tauberian conditions.*
If $R = 0$ in Theorem 3.5, then there are no moment conditions on ψ_1.

3.1.4 Boundedness of the Peetre maximal operator

We will present a theorem which describes the boundedness of the Peetre maximal operator. We use the same notation introduced at the beginning of the last subsection. Especially, we have the functions $\psi_k \in \mathcal{S}(\mathbb{R}^n)$ and $\Psi_k = \hat{\psi}_k \in \mathcal{S}(\mathbb{R}^n)$ for all $k \in \mathbb{N}_0$.

Theorem 3.7: *Let* $\{w_k\}_{k\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $a, s \in \mathbb{R}$ *and* $0 < p, q \leq \infty$. *For some* $\varepsilon > 0$ *we assume* $\psi_0, \psi_1 \in \mathcal{S}(\mathbb{R}^n)$ *with*

$$|\psi_0| > 0 \quad on \quad \{x \in \mathbb{R}^n : |x| < \varepsilon\} \tag{3.22}$$

$$|\psi_1| > 0 \quad on \quad \{x \in \mathbb{R}^n : \varepsilon/2 < |x| < 2\varepsilon\}\ . \tag{3.23}$$

If $a > \frac{n}{p} + \alpha$, *then*

$$\left\| 2^{ks}(\Psi_k^* f)_a w_k \right| \ell_q(L_p)\right\| \leq c \left\| 2^{ks}(\Psi_k * f) w_k \right| \ell_q(L_p)\right\| \tag{3.24}$$

holds for all $f \in \mathcal{S}'(\mathbb{R}^n)$.

Proof: As in the last proof we find the functions $\{\lambda_j\}_{j\in\mathbb{N}_0}$ with the properties (3.16)-(3.17) and

$$\sum_{k=0}^{\infty} \lambda_k(2^{-\nu}x)\psi_k(2^{-\nu}x) = 1 \quad \text{for all } \nu \in \mathbb{N}_0\ . \tag{3.25}$$

Instead of (3.18) we get the identity

$$\Psi_\nu * f = \sum_{k=0}^{\infty} \Lambda_{k,\nu} * \Psi_{k,\nu} * \Psi_\nu * f\ , \tag{3.26}$$

where

$$\Lambda_{k,\nu}(\xi) = [\lambda_k(2^{-\nu}\cdot)]^\wedge(\xi) = 2^{\nu n}\Lambda_k(2^\nu \xi) \qquad \text{for all } \nu, k \in \mathbb{N}_0\ .$$

The $\Psi_{k,\nu}$ are defined similarly. For $k \geq 1$ and $\nu \in \mathbb{N}_0$ we have $\Psi_{k,\nu} = \Psi_{k+\nu}$ and with the notation

$$\sigma_{k,\nu}(x) = \begin{cases} \psi_0(2^{-\nu}x) & \text{, if } k = 0 \\ \psi_\nu(x) & \text{, otherwise} \end{cases}$$

we get $\psi_k(2^{-\nu}x)\psi_\nu(x) = \sigma_{k,\nu}(x)\psi_{k+\nu}(x)$. Hence, we can rewrite (3.26) as

$$\Psi_\nu * f = \sum_{k=0}^{\infty} \Lambda_{k,\nu} * \hat{\sigma}_{k,\nu} * \Psi_{k+\nu} * f\ . \tag{3.27}$$

For $k \geq 1$ we get from Lemma 3.1

$$|(\Lambda_{k,\nu} * \hat{\sigma}_{k,\nu})(z)| = 2^{\nu n}|(\Lambda_k * \Psi)(2^\nu z)| \leq C_M 2^{\nu n}\frac{2^{-kM}}{(1+|2^\nu z|^a)} \tag{3.28}$$

for all $k, \nu \in \mathbb{N}_0$ and arbitrary large $M \in \mathbb{N}$. For $k = 0$ we get the estimate (3.28) by using Lemma 3.1 with $M = 0$. This together with (3.27) gives us

$$|(\Psi_\nu * f)(y)| \leq C_M 2^{\nu n}\sum_{k=0}^{\infty}\int\limits_{\mathbb{R}^n}\frac{2^{-kM}}{(1+|2^\nu(y-z)|^a)}|(\Psi_{k+\nu} * f)(z)|dz \ . \tag{3.29}$$

For fixed $r \in (0,1]$ we divide both sides of (3.29) by $(1+|2^\nu(x-y)|^a)$ and we take the supremum with respect to $y \in \mathbb{R}^n$. Using the inequalities

$$(1+|2^\nu(y-z)|^a)(1+|2^\nu(x-y)|^a) \geq c(1+|2^\nu(x-z)|^a) \ ,$$
$$|(\Psi_{k+\nu} * f)(z)| \leq |(\Psi_{k+\nu} * f)(z)|^r(\Psi^*_{k+\nu}f)_a(x)^{1-r}(1+|2^{k+\nu}(x-y)|^a)^{1-r}$$

and

$$\frac{(1+|2^{k+\nu}(x-z)|^a)^{1-r}}{(1+|2^\nu(x-y)|^a)} \leq \frac{2^{ka}}{(1+|2^{k+\nu}(x-y)|^a)^r} \ ,$$

we get

$$(\Psi^*_\nu f)_a(x) \leq C_M\sum_{k=0}^{\infty}2^{-k(M+n-a)}(\Psi^*_{k+\nu}f)_a(x)^{1-r}\int\limits_{\mathbb{R}^n}\frac{2^{(k+\nu)n}|(\Psi_{k+\nu} * f)(z)|^r}{(1+|2^{k+\nu}(x-y)|^a)^r}dz \ . \tag{3.30}$$

Now, we apply Lemma 3.4 with

$$\gamma_\nu = (\Psi^*_\nu f)_a(x) \ , \qquad \beta_\nu = \int\limits_{\mathbb{R}^n}\frac{2^{\nu n}|(\Psi_\nu * f)(z)|^r}{(1+|2^\nu(x-y)|^a)^r}dz \ , \quad \nu \in \mathbb{N}_0$$

$N = M + n - a$, $C_N = C_M + n - a$ and N^0 in (3.8) equals the order of the distribution $f \in \mathcal{S}'(\mathbb{R}^n)$.
By Lemma 3.4 we obtain for every $N \in \mathbb{N}$, $x \in \mathbb{R}^n$ and $\nu \in \mathbb{N}_0$

$$(\Psi^*_\nu f)_a(x)^r \leq C_N\sum_{k=0}^{\infty}2^{-kNr}\int\limits_{\mathbb{R}^n}\frac{2^{(k+\nu)n}|(\Psi_{k+\nu} * f)(z)|^r}{(1+|2^{k+\nu}(x-y)|^a)^r}dz \ . \tag{3.31}$$

We point out that (3.31) holds also for $r > 1$, where the proof is much simpler. We only have to take (3.29) with $a+n$ instead of a, divide both sides by $(1+|2^\nu(x-y)|^a)$ and apply Hölder's inequality with respect to k and then z.
Multiplying (3.31) by $w_\nu(x)^r$ we derive with the properties of our weight sequence

$$(\Psi^*_\nu f)_a(x)^r w_\nu(x)^r \leq C'_N\sum_{k=0}^{\infty}2^{-k(N-\alpha_1)r}\int\limits_{\mathbb{R}^n}\frac{2^{(k+\nu)n}|(\Psi_{k+\nu} * f)(z)|^r w_{k+\nu}(z)^r}{(1+|2^{k+\nu}(x-y)|^{a-\alpha})^r}dz \ , \tag{3.32}$$

for all $x \in \mathbb{R}^n$, $\nu \in \mathbb{N}_0$ and all $N \in \mathbb{N}$.
Now, choosing $r > 0$ with $\frac{n}{a-\alpha} < r < p$ the function

$$\frac{1}{(1+|z|)^{r(a-\alpha)}} \in L_1(\mathbb{R}^n)$$

and by the majorant property of the Hardy-Littlewood maximal operator (see [StWe71], Chapter 2) it follows

$$(\Psi_\nu^* f)_a(x)^r w_\nu(x)^r \leq C'_N \sum_{k=0}^{\infty} 2^{-k(N-\alpha_1)r} M(|\Psi_{k+\nu} * f|^r w_{k+\nu}^r)(x) \ . \tag{3.33}$$

We choose $N > 0$ such that $N > -s + \alpha_1$ and denote

$$g_k(x) = 2^{krs} M(|\Psi_k * f|^r w_k^r)(x) \ .$$

From (3.33) we derive

$$G_\nu(x) = (\Psi_\nu^* f)_a(x)^r w_\nu(x)^r \leq C \sum_{k \geq \nu}^{\infty} 2^{-k(N-\alpha_1)r} g_k(x) \ .$$

So, for $0 < \delta < N + s - \alpha_1$, we can apply Lemma 3.3 with the $\ell_{q/r}(L_{p/r})$ norm. This gives us

$$\left\| 2^{krs} (\Psi_k^* f)_a(x)^r w_k(x)^r \right| \ell_{q/r}(L_{p/r}) \Big\| \leq c \left\| 2^{krs} M(|\Psi_k * f|^r w_k^r)(x) \right| \ell_{q/r}(L_{p/r}) \Big\| \tag{3.34}$$

Rewriting the left hand side of (3.34) and using the scalar Hardy-Littlewood theorem [FeS71] (we recall $r < p$) on the right hand side, we finally get

$$\left\| 2^{ks} (\Psi_k^* f)_a w_k \right| \ell_q(L_p) \Big\| \leq c \left\| 2^{ks} (\Psi_k * f) w_k \right| \ell_q(L_p) \Big\| \ ,$$

and the proof is complete. □

3.2 Local means characterization

In this section we combine the two previous subsections to derive a generalization of the local means characterization as in [Tri92] and [Ry99] for the unweighted Besov spaces. The Peetre maximal operator was defined in Section 3.1.1 and the functions ψ_0, ψ_1 belong to $\mathcal{S}(\mathbb{R}^n)$.

Theorem 3.8: *Let $\boldsymbol{w} = \{w_k\}_{k \in \mathbb{N}_0} \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$, $0 < p, q \leq \infty$ and let $s, a \in \mathbb{R}$, $R \in \mathbb{N}_0$ with $a > \frac{n}{p} + \alpha$ and $R > s + \alpha_2$. If*

$$D^\beta \psi_1(0) = 0 \ , \quad \textit{for } 0 \leq |\beta| < R \ , \tag{3.35}$$

and

$$|\psi_0(x)| > 0 \quad \textit{on} \quad \{x \in \mathbb{R}^n : |x| < \varepsilon\} \tag{3.36}$$

$$|\psi_1(x)| > 0 \quad \textit{on} \quad \{x \in \mathbb{R}^n : \varepsilon/2 < |x| < 2\varepsilon\} \tag{3.37}$$

for some $\varepsilon > 0$*, then*

$$\left\| f \middle| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right\| \sim \left\| 2^{ks}(\Psi_k * f) w_k \middle| \ell_q(L_p) \right\| \sim \left\| 2^{ks}(\Psi^*_k f)_a w_k \middle| \ell_q(L_p) \right\|$$

holds for all $f \in \mathcal{S}'(\mathbb{R}^n)$*.*

Remark 3.9:

(a) The proof of Theorem 3.8 follows immediately from Theorem 3.5 and Theorem 3.7.

(b) If $R = 0$, then there are no moment conditions (3.35) on ψ_1.

Next we reformulate the Theorem 3.8 in the sense of [Tri92, Subsection 2.5.3].

Let $B = \{x \in \mathbb{R}^n : |x| < 1\}$ be the unit ball and $k \in \mathcal{S}(\mathbb{R}^n)$ a function with support in B. For a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ the corresponding local means are defined by (at least formally)

$$k(t,f)(x) = \int_{\mathbb{R}^n} k(y) f(x+ty) dy = t^{-n} \int_{\mathbb{R}^n} k\left(\frac{y-x}{t}\right) f(y) dy \ , \quad x \in \mathbb{R}^n, t > 0 \ . \tag{3.38}$$

Let $k_0, k^0 \in \mathcal{S}(\mathbb{R}^n)$ be two functions satisfying

$$\operatorname{supp} k_0 \subseteq B \ , \qquad\qquad \operatorname{supp} k^0 \subseteq B \ , \tag{3.39}$$

and

$$\hat{k}_0(0) \neq 0 \ , \qquad\qquad \hat{k^0}(0) \neq 0 \ . \tag{3.40}$$

For $N \in \mathbb{N}_0$ we define the iterated Laplacian

$$k(y) := \Delta^N k^0(y) = \left(\sum_{j=1}^n \frac{\partial^2}{\partial y_j^2} \right)^N k^0(y) \ , \qquad y \in \mathbb{R}^n \ . \tag{3.41}$$

It follows easily that

$$\check{k}(x) = |x|^{2N} \check{k^0}(x) \qquad \text{and that implies} \tag{3.42}$$

$$D^\beta \check{k}(0) = 0 \qquad \text{for} \qquad 0 \leq |\beta| < 2N \ . \tag{3.43}$$

Using this notation we come to the usual local means characterization.

Theorem 3.10: *Let* $\boldsymbol{w} = \{w_j\}_{j \in \mathbb{N}_0} \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$*,* $0 < p, q \leq \infty$*,* $s \in \mathbb{R}$*. Furthermore, let* $N \in \mathbb{N}_0$ *with* $2N > s + \alpha_2$ *and let* $k_0, k^0 \in \mathcal{S}(\mathbb{R}^n)$ *and the function* k *be defined as above. Then*

$$\| k_0(1,f) w_0 | L_p(\mathbb{R}^n) \| + \left(\sum_{j=1}^\infty 2^{jsq} \left\| k(2^{-j}, f) w_j \middle| L_p(\mathbb{R}^n) \right\|^q \right)^{1/q} \sim \left\| f \middle| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right\|$$

holds for all $f \in \mathcal{S}'(\mathbb{R}^n)$*.*

Proof: We put

$$\psi_0 = k_0^\vee \ , \qquad \psi_1 = k^\vee(\cdot/2) \ .$$

Then the Tauberian conditions (3.36) and (3.37) are satisfied and due to (3.43) also the moment conditions (3.35) are fulfilled. If we define ψ_j for $j \in \mathbb{N}_0$ as in (3.10), then we get

$$(\psi_j \hat{f})^\vee(x) = c(\psi_j^\vee * f)(x) = c \int\limits_{\mathbb{R}^n} (\mathcal{F}\psi_j)(y) f(x+y) dy \ . \tag{3.44}$$

For $j = 0$ we get $\mathcal{F}\psi_0 = k_0$ and for $j \geq 1$ we obtain

$$(\mathcal{F}\psi_j)(y) = (\mathcal{F}\psi_1(2^{-j+1}\cdot))(y) = 2^{(j-1)n}(\mathcal{F}\psi_1)(2^{j-1}y) = 2^{jn}k(2^j y) \ .$$

This and the equation (3.44) lead to

$$(\psi_j \hat{f})^\vee(x) = c2^{jn} \int\limits_{\mathbb{R}^n} k(2^j y) f(x+y) dy = ck(2^{-j}, f)(x) \ , \quad j \in \mathbb{N}_0 \ , \quad x \in \mathbb{R}^n \ .$$

Together with Theorem 3.8 the proof is complete. □

Remark 3.11: If we take $w_j \equiv 1$ for all $j \in \mathbb{N}_0$, we obtain the local means characterization for the usual Besov spaces. If we now compare our result with Section 2.5.3 in [Tri92], we get an improvement with respect to $N \in \mathbb{N}_0$. The condition in [Tri92] is $2N > \max(s, \sigma_p)$ where $\sigma_p = \max(0, n(1/p-1))$. We derived $2N > s$ in Theorem 3.10 ($\alpha_2 = 0$ for $w_j \equiv 1$) which seems to be more natural (see also Remark 1.11 in [Tri06] for a short history on this condition).
Furthermore, we proved the equivalence of the (quasi-)norms for all $f \in \mathcal{S}'(\mathbb{R}^n)$ by this method where in [Tri92] the equivalence does only hold for $f \in B^s_{pq}(\mathbb{R}^n)$.

For the last modification of the local means representation we introduce some necessary notation. For $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$ we denote by $Q_{\nu m}$ the cube centered at the point $2^{-\nu}m = (2^{-\nu}m_1, \ldots, 2^{-\nu}m_n)$ with sides parallel to coordinate axes and of length $2^{-\nu}$. Hence

$$Q_{\nu m} = \{x \in \mathbb{R}^n : |x_i - 2^{-\nu}m_i| \leq 2^{-\nu-1}, i = 1, \ldots, n\} \ , \qquad \nu \in \mathbb{N}_0 \ , m \in \mathbb{Z}^n \ . \tag{3.45}$$

If $\gamma > 0$, then $\gamma Q_{\nu m}$ denotes a cube concentric with $Q_{\nu m}$ with sides also parallel to coordinate axes and of length $\gamma 2^{-\nu}$.
Defining the Peetre maximal operator by (3.3), we get

$$(\Psi_\nu^* f)_a(x) \geq c \sup_{x-y \in \gamma Q_{\nu m}} |(\Psi_\nu * f)(y)| \ , \quad \nu \in \mathbb{N}_0 \ , x \in \mathbb{R}^n \ ,$$

where the constant c only depends on $a > 0$, $\gamma > 0$ and does not depend on $x \in \mathbb{R}^n$ and $\nu \in \mathbb{N}_0$.
With this simple observation we get immediately the following conclusion of Theorem 3.8.

Theorem 3.12: *Let* $\boldsymbol{w} = \{w_k\}_{k\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$*,* $0 < p, q \le \infty$*,* $s \in \mathbb{R}$*. For* $N \in \mathbb{N}_0$ *with* $2N > s + \alpha_2$ *let* k_0, k^0, k *be as in Theorem 3.10. Then for every* $\gamma > 0$

$$\left\| f \,|\, B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right\| \sim \left\| \sup_{(x-y)\in\gamma Q_{0,0}} |k_0(1,f)(y)| \,\middle|\, L_p(\mathbb{R}^n, w_0) \right\| + \left(\sum_{j=1}^{\infty} 2^{jsq} \left\| \sup_{(x-y)\in\gamma Q_{j,0}} |k(2^{-j},f)(y)| \,\middle|\, L_p(\mathbb{R}^n, w_j) \right\|^q \right)^{1/q}, \tag{3.46}$$

holds for all $f \in \mathcal{S}'(\mathbb{R}^n)$*.*

4 Application of the local means characterization

In this chapter we apply the local means characterization to derive two results on pointwise multipliers and the invariance under diffeomorphisms, which are well known in the classical Besov spaces cases ([Tri92, Sections 4.2 and 4.3]).

4.1 Pointwise multipliers

Let g be a bounded function on $\mathbb{R}^n$. We ask, under which conditions the mapping $f \mapsto gf$ makes sense and generates a bounded operator in a given space $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$. First of all, we have to specify what is meant by gf since at first glance $f \in \mathcal{S}'(\mathbb{R}^n)$ and is therefore not defined pointwise.

Remark 4.1: The interpretation of $g \cdot f$ is a bit sophisticated. We approximate f and g by smooth functions, f_j and g_j. The limit of $g_j \cdot f_j$ exists in $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$, see [Tri92, Remark 1/4.2.2], and we define $g \cdot f = \lim_{j\to\infty} g_j \cdot f_j$, where $g_j \cdot f_j$ has to be understood in the usual pointwise sense, as limit element. For a more detailed discussion of this procedure we refer also to [RuSi96, Chapter 4].

We follow closely [Tri92, 4.2.2] and adapt the proofs to our situation. First, we prove a lemma which is important for pointwise multipliers.

Lemma 4.2: *Let $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ and let $0 < p, q \le \infty$. Then for $s > \frac{n}{p} + \alpha + \alpha_1$ and all $\gamma > 0$ there is a constant $c_\gamma > 0$ such that*

$$\left\| w_0(\cdot) \sup_{|\cdot - y| \le \gamma} |f(y)| \,\middle|\, L_p(\mathbb{R}^n) \right\| \le c_\gamma \left\| f \,\middle|\, B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right\| \quad \textit{holds for all } f \in B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}).$$

Proof: Let $\{\varphi_j\}_{j\in\mathbb{N}_0} \in \Phi(\mathbb{R}^n)$ be the chosen resolution of unity from the beginning of the previous chapter. Then we get for arbitrary $\varepsilon > 0$

$$\left\| w_0(\cdot) \sup_{|\cdot - y| \le \gamma} |f(y)| \,\middle|\, L_p(\mathbb{R}^n) \right\| \le c \sum_{j=0}^{\infty} 2^{j\varepsilon} \left\| w_0(\cdot) \sup_{|\cdot - y| \le \gamma} |(\varphi_j \hat{f})^\vee(y)| \,\middle|\, L_p(\mathbb{R}^n) \right\| .$$

For all $a > 0$ we have

$$\sup_{|x-y|\le\gamma} |(\varphi_j \hat{f})^\vee(y)| \le c 2^{ja} \sup_{z\in\mathbb{R}^n} \frac{|(\varphi_j \hat{f})^\vee|(x-z)}{1 + |2^j z|^a}$$

where the constant only depends on $\gamma > 0$. Using the property (2.4) of the weight sequence and Theorem 3.8, we obtain for arbitrary $a > n/p + \alpha$ and $\varepsilon > 0$

$$\begin{aligned}\left\| w_0(\cdot) \sup_{|\cdot - y| \le \gamma} |f(y)| \,\Big|\, L_p(\mathbb{R}^n) \right\| &\le c \left\| (\varphi_j^* f)_a \,\Big|\, B_{p1}^{a+\alpha_1+\varepsilon,mloc}(\mathbb{R}^n, \boldsymbol{w}) \right\| \\ &\le c' \left\| f \,\big|\, B_{p1}^{a+\alpha_1+\varepsilon,mloc}(\mathbb{R}^n, \boldsymbol{w}) \right\| \\ &\le c'' \left\| f \,\big|\, B_{pq}^{s,mloc}(\mathbb{R}^n, \boldsymbol{w}) \right\| ,\end{aligned}$$

for $s > \frac{n}{p} + \alpha + \alpha_1$ and $f \in \mathcal{S}'(\mathbb{R}^n)$. $\square$

Let k_0, $k \in \mathcal{S}(\mathbb{R}^n)$ and $k(t, f)$ be the same functions as in (3.38)-(3.41). For $g \in C^m(\mathbb{R}^n)$ we study gf where $f \in B_{pq}^{s,mloc}(\mathbb{R}^n, \boldsymbol{w})$. First, we prove the theorem and after that we discuss, how gf has to be understood.

Theorem 4.3: *Let $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $s \in \mathbb{R}$ and let $0 < p, q \le \infty$. If $m \in \mathbb{N}$ is sufficiently large, then there exists a positive number c_m such that*

$$\left\| gf \,\big|\, B_{pq}^{s,mloc}(\mathbb{R}^n, \boldsymbol{w}) \right\| \le c_m \sum_{|\beta| \le m} \left\| D^\beta g \,\big|\, L_\infty(\mathbb{R}^n) \right\| \left\| f \,\big|\, B_{pq}^{s,mloc}(\mathbb{R}^n, \boldsymbol{w}) \right\| \tag{4.1}$$

for all $g \in C^m(\mathbb{R}^n)$ and all $f \in B_{pq}^{s,mloc}(\mathbb{R}^n, \boldsymbol{w})$.

Proof: First Step: Firstly, we prove the theorem under the additional assumption $s > \frac{n}{p} + \alpha + \alpha_1$. We use the Taylor expansion of $g \in C^m(\mathbb{R}^n)$

$$g(x) = \sum_{|\beta| \le m-1} \frac{D^\beta g(y)}{\beta!} (x-y)^\beta + \sum_{|\beta| = m} \frac{D^\beta g(y + \theta(x-y))}{\beta!} (x-y)^\beta , \tag{4.2}$$

for $\theta \in (0, 1)$. By (3.38) we have

$$\begin{aligned} k(2^{-j}, f)(x) &= \int_{\mathbb{R}^n} k(y) f(x + 2^{-j}y) g(x + 2^{-j}y) dy \\ &= \sum_{|\beta| \le m-1} \frac{D^\beta g(x)}{\beta!} 2^{-j|\beta|} \int_{\mathbb{R}^n} y^\beta k(y) f(x + 2^{-j}y) dy + 2^{-jm} \int_{\mathbb{R}^n} k(y) r_m(x, 2^{-j}, y) f(x + 2^{-j}y) dy ,\end{aligned}$$

where the remainder term in Taylor's expansion, $r_m(x, 2^{-j}, y)$, is in any case uniformly bounded. If we choose $N \in \mathbb{N}_0$ in (3.41) sufficiently large, for each $|\beta| \le m - 1$ the function $k_\beta(y) = y^\beta k(y)$ is a new kernel for which Theorem 3.10 holds. Thus, choosing $m > s + \alpha_2$ and using Theorem 3.10 for every $|\beta| \le m - 1$ we obtain

$$\begin{aligned}\left(\sum_{j=1}^{\infty} 2^{jsq} \left\| w_j k(2^{-j}, f) \,\big|\, L_p(\mathbb{R}^n) \right\|^q \right)^{1/q} &\le c \sum_{|\beta| \le m-1} \left\| D^\beta g \,\big|\, L_\infty(\mathbb{R}^n) \right\| \left\| f \,\big|\, B_{pq}^{s,mloc}(\mathbb{R}^n, \boldsymbol{w}) \right\| \\ &+ c \sum_{|\beta| \le m} \left\| D^\beta g \,\big|\, L_\infty(\mathbb{R}^n) \right\| \left\| w_0(\cdot) \sup_{|\cdot - y| \le 1} |f(y)| \,\Big|\, L_p(\mathbb{R}^n) \right\| .\end{aligned}$$

Now, Lemma 4.2 with $\gamma = 1$ proves the theorem provided $s > \frac{n}{p} + \alpha + \alpha_1$.
Second Step: Let $-\infty < s \leq \frac{n}{p} + \alpha + \alpha_1$ and let $l \in \mathbb{N}$ with $s + 2l > \frac{n}{p} + \alpha + \alpha_1$. By lift property (see Section 2.4) any $f \in B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ can be represented as $f = (\mathrm{id} + (-\Delta)^l)h$, with

$$h \in B^{s+2l,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \quad \text{and} \quad \left\| h \right| B^{s+2l,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \big\| \sim \left\| f \right| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \big\| \ . \tag{4.3}$$

We have

$$gf = (\mathrm{id} + (-\Delta)^l)gh + \sum_{|\beta| < 2l} D^\beta (g_\beta h) \ ,$$

where each g_β is a sum of terms of the type $D^\gamma g$ with $|\gamma| \leq 2l$. Now, Theorem 2.20 shows

$$\left\| gf \right| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \big\| \leq c \sum_{|\beta| \leq 2l} \left\| g_\beta h \right| B^{s+2l,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \big\| \ .$$

If $l \in \mathbb{N}$ is sufficiently large, which is $m - 2l > s + 2l + \alpha_2$, we can apply the first step and obtain

$$\left\| gf \right| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \big\| \leq c \sum_{|\beta| \leq m} \left\| D^\beta g \right| L_\infty(\mathbb{R}^n) \big\| \left\| h \right| B^{s+2l,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \big\| \ .$$

Finally, (4.3) proves the theorem. □

Remark 4.4: The Theorem above was proved in the special case of 2-microlocal spaces $C^{s,s'}_{x_0}(\mathbb{R}^n)$ by Meyer in [Mey97, Lemma 3.3] with the help of para-products. For the spaces $B^{s,s'}_{pq}(\mathbb{R}^n, 0)$ with $p, q \geq 1$ it has been proved by Xu in [Xu96, Theorem 3.1].

4.2 Invariance under diffeomorphisms

In this section we show that the spaces $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ are invariant under diffeomorphisms. The result and the proof are closely related to Section 4.3 in[Tri92]. Let $m \in \mathbb{N}$, then we call an isomorphism $\psi : \mathbb{R}^n \to \mathbb{R}^n$ an m-diffeomorphism if the components $\psi_j(x)$ of $\psi(x) = (\psi_1(x), \ldots, \psi_n(x))$ have classical continuous derivatives up to the order k and the functions $D^\beta \psi_j(x)$ are bounded for all $0 < |\beta| \leq m$, $1 \leq j \leq n$ and all $x \in \mathbb{R}^n$. Furthermore, the Jacobian matrix ψ_* has to fulfill $|\det \psi_*(x)| \geq d > 0$ for all $x \in \mathbb{R}^n$. If $y = \psi(x)$ is a m-diffeomorphism for every $m \in \mathbb{N}$, then it is called diffeomorphism.
We want to prove that $f \to f \circ \psi$ is a linear and bounded operator in all spaces $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$. If ψ is a diffeomorphism, then

$$f \circ \psi(x) = f(\psi(x)) \tag{4.4}$$

makes sense for all $f \in \mathcal{S}'(\mathbb{R}^n)$. If ψ is only an m-diffeomorphism, then (4.4) has to be understood as an approximation procedure with smooth functions (see also Remark 4.1). In the proof we use the local means characterization in the form of Theorem 3.10.

First of all, we have to prove two lemmas which will be useful later on.
We need a modification of Theorem 3.10. Therefore, let k_0 and k^0 be kernels in the sense of (3.39)-(3.41) with $N \in \mathbb{N}_0$ large enough and $a(x)$ be an $n \times n$ matrix with real-valued continuous entries $a_{ij}(x)$, where $x \in \mathbb{R}^n$ and $i, j \in \{1, \dots, n\}$. Further, there exist two numbers $d, d' > 0$ with

$$|a_{ij}(x)| \leq d' \qquad \text{for all } x \in \mathbb{R}^n,\ i, j \in \{1, \dots, n\} \text{ and} \tag{4.5}$$

$$|\det a(x)| \geq d > 0 \qquad \text{for all } x \in \mathbb{R}^n. \tag{4.6}$$

Since, $y \mapsto ya(x)$ is an isomorphic mapping for fixed $x \in \mathbb{R}^n$ we can generalize (3.38) by

$$k(a, t, f)(x) = \int_{\mathbb{R}^n} k(y) f(x + ta(x)y) dy\ . \tag{4.7}$$

Lemma 4.5: *Let $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $s \in \mathbb{R}$ and let $0 < p, q \leq \infty$. Further, let $a(x)$ be the above matrix with* (4.5), (4.6) *and let k_0 and k be the functions from* (3.39)-(3.41). *Then there exists a constant c such that*

$$\| k_0(a, 1, f) w_0 |\, L_p(\mathbb{R}^n)\| + \left(\sum_{j=1}^{\infty} 2^{jsq} \left\| k(a, 2^{-j}, f) w_j \right| L_p(\mathbb{R}^n) \right\|^q \right)^{1/q} \leq c \left\| f \right| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right\| \tag{4.8}$$

holds for all $f \in \mathcal{S}'(\mathbb{R}^n)$.

Proof: Let B be the collection of all matrices $b = \{b_{ij}\}_{i,j=1}^n$ satisfying (4.5) and (4.6). For fixed $b \in B$ we derive by this properties

$$k(b, t, f)(x) = k^b(t, f)(x) \qquad \text{whereas} \quad k^b(y) = ck(b^{-1}y) \tag{4.9}$$

is a modified kernel in the sense of (3.39)-(3.41). The same holds for k_0^b so that we can apply now Theorem 3.10 with the new kernels, and get

$$\left\| k_0^b(1, f) w_0 \right| L_p(\mathbb{R}^n) \right\| + \left(\sum_{j=1}^{\infty} 2^{jsq} \left\| k^b(2^{-j}, f) w_j \right| L_p(\mathbb{R}^n) \right\|^q \right)^{1/q} \sim c \left\| f \right| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right\|$$

for all $f \in \mathcal{S}'(\mathbb{R}^n)$. Now, we obtain (4.8) from this formula in taking the supremum over all $b \in B$ inside the L_p quasi-norms. $\square$

To get the invariance under diffeomorphisms of our spaces we also need a special restriction on the diffeomorphisms. From now on we consider only diffeomorphisms ψ which satisfy for a given weight sequence $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ that $w_0 \circ \psi \sim w_0$. That means, there exist $c_1, c_2 > 0$ such that $c_1 w_0(x) \leq (w_0 \circ \psi)(x) \leq c_2 w_0(x)$ holds for all $x \in \mathbb{R}^n$.
Now, the main theorem can be stated.

Theorem 4.6: *Let $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $0 < p, q \leq \infty$ and let $s \in \mathbb{R}^n$. Further, let ψ be a m-diffeomorphism for $m \in \mathbb{N}$ large enough and with $w_0 \circ \psi \sim w_0$. Then $f \mapsto f \circ \psi$ is an isomorphic mapping from $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ onto $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$.*

Proof: First Step: It is enough to prove that there exists a constant $c > 0$ such that

$$\left\| f \circ \psi \right| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \big\| \leq c \left\| f \right| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \big\| \qquad \text{for all } f \in \mathcal{S}'(\mathbb{R}^n). \tag{4.10}$$

The reverse inequality follows immediately if we use ψ^{-1} in (4.10) with $w_0 \circ \psi^{-1} \sim w_0$. Furthermore, we always assume that f is a smooth function.
Second Step: Let $s > \frac{n}{p} + \alpha + 2\alpha_1 + \alpha + 2$, then we can find a number $K \in \mathbb{N}$ with

$$\alpha_1 + \alpha_2 + 1 < K + \frac{n}{p} + \alpha + \alpha_1 < s \qquad \text{and} \qquad s + \alpha_2 < 2K \ . \tag{4.11}$$

We use the local means characterization, Theorem 3.10, with some kernels k_0, k and $N \in \mathbb{N}_0$ large enough. To simplify our notation we write $k(1, f) := k_0(1, f)$ and we put the first summand with k_0 and w_0 into the infinite summation with $j = 0$. So we get with this notation

$$\begin{aligned} \left\| f \circ \psi \right| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \big\| &\leq c \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| w_j k(2^{-j}, f \circ \psi) \right| L_p(\mathbb{R}^n) \big\|^q \right)^{1/q} \\ &\leq c \left(\sum_{j=0}^{\infty} 2^{jq(s+\alpha_2)} \left\| w_0(x) \int_{\mathbb{R}^n} k(y) f(\psi(x + 2^{-j}y)) dy \right| L_p(\mathbb{R}^n) \right\|^q \right)^{1/q} . \end{aligned} \tag{4.12}$$

We use Taylor expansion on ψ and obtain

$$\psi(x + 2^{-j}y) = \psi(x) + 2^{-j}\psi_*(x) \cdot y + \sum_{2 \leq |\beta| < 2K} 2^{-j|\beta|} \frac{D^\beta \psi(x)}{\beta!} y^\beta + 2^{-2Kj} R_{2K}(x, 2^{-j}, y) \ ,$$

where $D^\beta \psi$ and the remainder term R_{2K} stand for appropriate vectors. Again, we apply Taylor expansion, now on f, and derive

$$\begin{aligned} & f \left[\psi(x) + 2^{-j}\psi_*(x) \cdot y + \sum_{2 \leq |\beta| < 2K} \cdot \cdot + 2^{-2Kj} R_{2K} \right] \\ &\quad = f \left[\psi(x) + 2^{-j}\psi_*(x) \cdot y + \sum_{2 \leq |\beta| < 2K} \cdot\cdot \right] + 2^{-2Kj} \widetilde{R_{2K}}(x, 2^{-j}, y) \cdot (\nabla f)(\xi) \ , \end{aligned} \tag{4.13}$$

where the last term is a scalar product with an immaterially modified remainder term. Now, putting the last summand of (4.13) into (4.12) and using $2K > s + \alpha_2$ we can estimate this by

$$c \left\| w_0(x) \sup_{|\psi(x) - z| < c'} |(\nabla f)(z)| \right| L_p(\mathbb{R}^n) \right\| \ .$$

An obvious substitution, $w_0(\psi^{-1}(x)) \le c w_0(x)$, Lemma 4.2 and Theorem 2.20 show that this is bounded by $c\left\| f | \, B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})\right\|$. To handle the first term in (4.13) we use Taylor again and get

$$f\left[\psi(x) + 2^{-j}\psi_*(x)\cdot y + \sum_{2\le|\beta|<2K} \cdot\cdot\right]$$
$$= \sum_{0\le|\gamma|<K} \frac{D^\gamma f(\psi(x)+2^{-j}\psi_*(x)\cdot y)}{\gamma!}\left(\sum_{2\le|\beta|<2K}\cdot\cdot\right)^\gamma + \sum_{|\gamma|=K}\frac{D^\gamma f}{\gamma!}\left(\sum_{2\le|\beta|<2K}\cdot\cdot\right)^\gamma . \tag{4.14}$$

From

$$\left|\left(\sum_{2\le|\beta|<2K}\cdot\cdot\right)^\gamma\right| \le c2^{-2Kj} \quad \text{for } |\gamma| = K \ ,$$

we can estimate the last term of (4.14) in (4.12) by

$$c\sum_{|\gamma|=K}\left\| w_0(x) \sup_{|\psi(x)-z|<c'} |D^\gamma f(z)| \, \Big| \, L_p(\mathbb{R}^n)\right\| \ .$$

The same substitution as above, $w_0(\psi^{-1}(x)) \le c w_0(x)$, Lemma 4.2 and Theorem 2.20 show the boundedness by $c\left\| f | \, B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})\right\|$ when $s - K > \frac{n}{p} + \alpha + \alpha_1$. Finally, it remains to estimate the first term of (4.14) in (4.12). The resulting term is

$$c\sum_{0\le|\gamma|<K}\left(\sum_{j=0}^\infty 2^{jsq}\left\| w_j(x)2^{-jb}\int_{\mathbb{R}^n} k(y)y^\delta D^\gamma f(\psi(x)+2^{-j}\psi_*(x)\cdot y)dy \, \Big| \, L_p(\mathbb{R}^n)\right\|^q\right)^{1/q} ,$$

where $b \ge 2|\gamma|$ and $|\delta| \le (2K-1)|\gamma|$. For large $N \in \mathbb{N}_0$ we get that $\widetilde{k_\gamma}(y) := k(y)y^\delta$ are new kernels in the sense of Theorem 3.10 and we can estimate

$$\le c' \sum_{0\le|\gamma|<K}\left(\sum_{j=0}^\infty 2^{jq(s+\alpha_2-b)}\left\| w_0(x)\widetilde{k_\gamma}(\psi_*\circ\psi^{-1}, 2^{-j}, D^\gamma f)(\psi(x)) \, \Big| \, L_p(\mathbb{R}^n)\right\|^q\right)^{1/q} .$$

We substitude and use $w_0(\psi^{-1}(x)) \le c w_0(x)$. Hence, we can apply Lemma 4.5 and we derive

$$\le c' \sum_{0\le|\gamma|<K}\left\| D^\gamma f | \, B^{s+\alpha_1+\alpha_2-b,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})\right\| \ .$$

This can be estimated by $c\left\| f | \, B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})\right\|$ if $K > \alpha_1 + \alpha_2 + 1$ and therefore $s > \frac{n}{p} + \alpha + 2\alpha_1 + \alpha_2 + 2$.

Third Step: Let $s \le \frac{n}{p} + \alpha + 2\alpha_1 + \alpha_2 + 2$ then there is an $l \in \mathbb{N}$ such that $s + 2l > \frac{n}{p} + \alpha + 2\alpha_1 + \alpha_2 + 2$. As in the previous section we present $f \in B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ by

$$f = (\mathrm{id} + (-\Delta)^l)h \quad h \in B^{s+2l,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \tag{4.15}$$

and

$$\left\| f \right| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})\big\| \sim \left\| h \right| B^{s+2l,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})\big\| \ . \tag{4.16}$$

We have

$$f(x) = \sum_{|\beta| \le 2l} c_\beta(x)(D^\beta h \circ \psi \circ \psi^{-1})(x) \ , \tag{4.17}$$

where c_β are some functions. We assume that they are smooth and that we can apply Theorem 4.3 and obtain

$$\left\| f \circ \psi \right| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})\big\| \le c \sum_{|\beta| \le 2l} \left\| D^\beta h \circ \psi \right| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})\big\| \le c' \left\| h \circ \psi \right| B^{s+2l,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})\big\| \ .$$

Finally, the second step and (4.15) lead to the result we focused on. $\square$

The question arises, what conditions on ψ are sufficient to have $w_0 \circ \psi \sim w_0$. One result which is independent of the chosen weight sequence is that ψ satisfies $\psi(x) = x$ for x near to infinity ($|x| > R$ for some $R > 0$).
This is stated in the following lemma.

Lemma 4.7: *Let w_0 be an admissible weight function. Let $R > 0$ and ψ be an m-diffeomorphism with $\psi(x) = x$ for $|x| > R$, then there exist constants $c_1, c_2 > 0$ such that*

$$c_1 w_0(x) \le (w_0 \circ \psi)(x) \le c_2 w_0(x) \qquad \text{holds for all } x \in \mathbb{R}^n. \tag{4.18}$$

Proof: If ψ is an m-diffeomorphism with the restriction above, then we define

$$a^* := \max_{1 \le i,j \le n} \sup_{x \in \mathbb{R}^n} \left| \frac{\partial \psi_i}{\partial x_j}(x) \right| \ . \tag{4.19}$$

Using the properties of the weight function w_0 and Taylor expansion of ψ we obtain

$$\begin{aligned} w_0(\psi(x)) &\le \mathrm{C} w_0(x)(1 + |x - \psi(x)|)^\alpha \le \mathrm{C} w_0(x)(1 + |x - \psi(0) - \psi_*(..) \cdot x|)^\alpha \\ &\le \mathrm{C} w_0(x) 2^\alpha (1 + |\psi(0)|)^\alpha (1 + |x - \psi_*(..) \cdot x|)^\alpha \\ &\le \mathrm{C}' w_0(x)(1 + |x - \psi_*(..) \cdot x|)^\alpha \ . \end{aligned}$$

Here $\psi_*(..)$ is the Jacobian where in every line different arguments from the line segment between 0 and x are possible. In every case, the absolute values from all entries of $\psi_*(..)$ are bounded by a^*. We can estimate from this property

$$|x - \psi_*(..) \cdot x| \le |x|(1 + a^* n) \qquad \text{for all } x \in \mathbb{R}^n. \tag{4.20}$$

Finally, we get from $\psi(x) = x$ for $|x| > R$ and the preceding calculation

$$w_0 \circ \psi(x) = w_0(\psi(x)) \le \begin{cases} C_{R,\alpha,\psi,n} w_0(x) & \text{for } |x| \le R \\ w_0(x) & \text{for } |x| > R \ . \end{cases} ,$$

Finally, to get the equivalence in (4.18) we use in the whole proof ψ^{-1} instead of ψ which also has the property $\psi^{-1}(x) = x$ for $|x| > R$. □

The restriction $\psi(x) = x$ for large x is not satisfactory. For the special case of the 2-microlocal Besov spaces $B^{s,s'}_{pq}(\mathbb{R}^n, x_0)$ with the weight sequence $w_j(x) = (1 + 2^j|x - x_0|)^{s'}$ a more moderate restriction on ψ can be used. Let us have a look on $w_0 \circ \psi$ for $s' \ge 0$, we have

$$w_0 \circ \psi(x) = w_0(\psi(x)) = (1 + |\psi(x) - x_0|)^{s'} \ .$$

Now, using Taylor expansion on ψ at the point x_0, we get

$$w_0 \circ \psi(x) = (1 + |\psi(x_0) + \psi_*(..) \cdot (x - x_0) - x_0|)^{s'} \ .$$

Finally, demanding $\psi(x_0) = x_0$ we obtain in the same manner as in (4.20)

$$= (1 + |\psi_*(..) \cdot (x - x_0)|)^{s'} \le C_{\psi,n,s'}(1 + |(x - x_0)|)^{s'} = C_{\psi,n,s'} w_0(x) \ .$$

Furthermore, the inverse ψ^{-1} trivially fulfills $\psi^{-1}(x_0) = x_0$ which leads to the result we aimed at. So the following corollary of Theorem 4.6 holds.

Corollary 4.8: *Let $x_0 \in \mathbb{R}^n$, $0 < p, q \le \infty$, $s \in \mathbb{R}$ and $s' \ge 0$, Further let ψ be an m-diffeomorphism with $m \in \mathbb{N}$ large enough and $\psi(x_0) = x_0$, then $f \mapsto f \circ \psi$ is an isomorphic mapping from $B^{s,s'}_{pq}(\mathbb{R}^n, x_0)$ onto $B^{s,s'}_{pq}(\mathbb{R}^n, x_0)$.*

Remark 4.9: Hong Xu presented in [Xu96, Theorem 5.2] this corollary in the case of $B^{s,s'}_{pq}(\mathbb{R}^n, 0)$ for $1 \le p, q \le \infty$ under the additional assumption $\psi(x) = x$ which seems to be superfluous.

5 Decompositions

In this chapter we present three decomposition theorems. We characterize the spaces $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ via decompositions in atoms, molecules and wavelets. First we introduce the basic notations.

5.1 Sequence spaces

We recall that for $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ we denote by $Q_{\nu m}$ the cube with center in $2^{-\nu}m$ with sides parallel to the axes with length $2^{-\nu}$. By $\chi_{\nu m}$ we denote the characteristic function of the cube $Q_{\nu m}$, defined by

$$\chi_{\nu m}(x) = \chi_{Q_{\nu m}}(x) = \begin{cases} 1 \,, & \text{if } x \in Q_{\nu m} \\ 0 \,, & \text{if } x \notin Q_{\nu m} \,. \end{cases} \tag{5.1}$$

Definition 5.1: *Let $\boldsymbol{w} = \{w_k\}_{k \in \mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $s \in \mathbb{R}$ and let $0 < p, q \leq \infty$. Then for all complex-valued sequences*

$$\lambda = \{\lambda_{\nu m} \in \mathbb{C} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$$

we define

$$b^{s,mloc}_{pq}(\boldsymbol{w}) = \left\{ \lambda : \big\| \lambda | \, b^{s,mloc}_{pq}(\boldsymbol{w}) \big\| := \left(\sum_{\nu=0}^{\infty} 2^{\nu(s-\frac{n}{p})q} \left(\sum_{m \in \mathbb{Z}^n} w^p_{\nu}(2^{-\nu}m) |\lambda_{\nu m}|^p \right)^{q/p} \right)^{1/q} \right\}$$

with the usual modifications if p or q are equal to infinity.

Remark 5.2: If one defines for given λ: $g_\nu(x) = \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi_{\nu m}(x)$, one obtains

$$\big\| \lambda | \, b^{s,mloc}_{pq}(\boldsymbol{w}) \big\| = \| 2^{\nu s} w_\nu g_\nu | \, \ell_q(L_p) \| \;.$$

5.2 Atomic and molecular decompositions

Atoms are the building blocks for the atomic decomposition. We say a function $f : \mathbb{R}^n \to \mathbb{C}$ belongs to $C^K(\mathbb{R}^n)$ if the function and all classical derivatives $D^\alpha f$ are continuous and bounded for $|\alpha| \leq K$.

Definition 5.3: *Let $K, L \in \mathbb{N}_0$ and let $\gamma > 1$. A function $a \in \mathbb{C}^K(\mathbb{R}^n)$ is called $[K, L]$-atom centered at $Q_{\nu m}$, if for some $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$*

$$\operatorname{supp} a \subseteq \gamma Q_{\nu m} \;, \tag{5.2}$$

$$|D^\beta a(x)| \leq 2^{|\beta|\nu} \;, \quad \textit{for } 0 \leq |\beta| \leq K \tag{5.3}$$

and

$$\int_{R^n} x^\beta a(x)dx = 0 \quad \text{if} \quad 0 \le |\beta| < L \text{ and } \nu \ge 1 \ . \tag{5.4}$$

Remark 5.4: If an atom a is centered at $Q_{\nu m}$, then we denote it by $a_{\nu m}$. We recall the definition $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$ and point out that for $\nu = 0$ or $L = 0$ there are no moment conditions (5.4) on the atoms.

Next, we define molecules, which are similar to atoms but they do not have compact supports.

Definition 5.5: *Let $K, L \in \mathbb{N}_0$ and let $M > 0$. A function $\mu \in C^K(\mathbb{R}^n)$ is called $[K, L, M]$-molecule concentrated in $Q_{\nu m}$, if for some $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$*

$$|D^\beta \mu(x)| \le 2^{|\beta|\nu}(1 + 2^\nu|x - 2^{-\nu}m|)^{-M} \ , \quad \text{for } 0 \le |\beta| \le K \tag{5.5}$$

and

$$\int_{R^n} x^\beta \mu(x)dx = 0 \quad \text{if} \quad 0 \le |\beta| < L \text{ and } \nu \ge 1 \ . \tag{5.6}$$

Remark 5.6: (a) For $L = 0$ or $\nu = 0$ there are no moment conditions on μ. If a molecule is concentrated in $Q_{\nu m}$, which means it satisfies (5.5), then it is denoted by $\mu_{\nu m}$.

(b) If $a_{\nu m}$ is a $[K, L]$-atom then it is a $[K, L, M]$-molecule for every $M > 0$.

First, we show the convergence of the molecular decomposition. The number σ_p was defined in (2.39).

Lemma 5.7: *Let $\boldsymbol{w} = \{w_k\}_{k\in\mathbb{N}_0} \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$ and let $0 < p, q \le \infty$, $s \in \mathbb{R}$. Furthermore, let $K, L \in \mathbb{N}_0$ and $M > 0$ with*

$$L > \sigma_p - s + \alpha_1 \qquad , \ K \text{ arbitrary and } M \text{ large enough} \ . \tag{5.7}$$

If $\lambda \in b^{s,mloc}_{pq}(\boldsymbol{w})$ and $\{\mu_{\nu m}\}_{\nu\mathbb{N}_0, m\in\mathbb{Z}^n}$ are $[K, L, M]$-molecules concentrated in $Q_{\nu m}$, then the sum

$$\sum_{\nu=0}^\infty \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\mu_{\nu m}(x) \tag{5.8}$$

converges in $\mathcal{S}'(\mathbb{R}^n)$.

Proof: We have to prove that the limit

$$\lim_{r\to\infty} \sum_{\nu=0}^r \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\mu_{\nu m}(x) \qquad \text{exists in } \mathcal{S}'(\mathbb{R}^n). \tag{5.9}$$

For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we get from the moment conditions (5.6) for fixed $\nu \in \mathbb{N}_0$

$$\begin{aligned}
&\int\limits_{\mathbb{R}^n} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\mu_{\nu m}(y)\varphi(y)dy \\
&\qquad = \int\limits_{\mathbb{R}^n} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\mu_{\nu m}(y)w_\nu(y)\times \\
&\qquad\qquad \times \left(\varphi(y) - \sum_{|\beta|<L} \frac{D^\beta\varphi(2^{-\nu}m)}{\beta!}(y-2^{-\nu}m)^\beta\right) w_\nu^{-1}(y)\frac{\langle y\rangle^\kappa}{\langle y\rangle^\kappa}dy \ ,
\end{aligned} \tag{5.10}$$

where $\kappa > 0$ will be specified later on. We use Taylor expansion of φ up to the order L and get with ξ on the line segment joining y and $2^{-\nu}m$

$$\varphi(y) = \sum_{|\beta|<L} \frac{D^\beta\varphi(2^{-\nu}m)}{\beta!}(y-2^{-\nu}m)^\beta + \sum_{|\beta|=L} \frac{D^\beta\varphi(\xi)}{\beta!}(y-2^{-\nu}m)^\beta \ .$$

In using the properties of the weight sequence and $\langle y\rangle^\kappa \le \langle y-2^{-\nu}m\rangle^\kappa\langle\xi\rangle^\kappa$, we estimate

$$\begin{aligned}
&|\mu_{\nu m}(y)| \left|\varphi(y) - \sum_{|\beta|<L} \frac{D^\beta\varphi(2^{-\nu}m)}{\beta!}(y-2^{-\nu}m)^\beta\right| w_\nu^{-1}(y)\frac{\langle y\rangle^\kappa}{\langle y\rangle^\kappa} \\
&\le c2^{-\nu(L-\alpha_1)}(1+2^\nu|y-2^{-\nu}m|)^{-M} \sum_{|\beta|=L} \frac{|D^\beta\varphi(\xi)|}{\beta!}|y-2^{-\nu}m|^L 2^{\nu L} w_0^{-1}(y)\frac{\langle\xi\rangle^\kappa\langle y-2^{-\nu}m\rangle^\kappa}{\langle y\rangle^\kappa} \\
&\le c' 2^{-\nu(L-\alpha_1)}(1+2^\nu|y-2^{-\nu}m|)^{L+\kappa-M}\langle y\rangle^{\alpha-\kappa} \sup_{\xi\in\mathbb{R}^n}\langle\xi\rangle^\kappa \sum_{|\beta|=L}\frac{|D^\beta\varphi(\xi)|}{\beta!} \ .
\end{aligned}$$

Hence, we derive from (5.10)

$$\begin{aligned}
\left|\int\limits_{\mathbb{R}^n} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\mu_{\nu m}(y)\varphi(y)dy\right| &\le c2^{-\nu(L-\alpha_1)} \sup_{x\in\mathbb{R}^n}\langle x\rangle^\kappa \sum_{|\beta|=L}\frac{|D^\beta\varphi(x)|}{\beta!}\times \\
&\quad \times \int\limits_{\mathbb{R}^n} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|w_\nu(y)(1+2^\nu|y-2^{-\nu}m|)^{L+\kappa-M}\langle y\rangle^{\alpha-\kappa}dy \ .
\end{aligned} \tag{5.11}$$

Now, let us suppose that $p \ge 1$. Using Hölder's inequality on the integral in (5.11) with $\kappa > \frac{n}{p'} + \alpha \ge n + \alpha$ we get

$$\begin{aligned}
&\int\limits_{\mathbb{R}^n} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|w_\nu(y)(1+2^\nu|x-2^{-\nu}m|)^{L+\kappa-M}\langle y\rangle^{\alpha-\kappa}dy \\
&\qquad\qquad \le c2^{-\nu(L+s-\alpha_1)}2^{\nu s}\left\|\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|(1+2^\nu|x-2^{-\nu}m|)^{L+\kappa-M}\right| L_p(\mathbb{R}^n,w_\nu)\right\| \ .
\end{aligned}$$

By choosing M large enough ($M > L + 2n + 2\alpha$) and using the same reasoning as in Lemma 5.15 (on page 55) with $j = \nu$ we get

$$\left| \int_{\mathbb{R}^n} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\mu_{\nu m}(y)\varphi(y)dy \right|$$
$$\leq c 2^{-\nu(L+s-\alpha_1)} 2^{\nu(s-\frac{n}{p})} \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^p w_\nu^p(2^{-\nu}m) \right)^{1/p} \sup_{x\in\mathbb{R}^n} \langle x\rangle^\kappa \sum_{|\beta|=L} \frac{|D^\beta\varphi(x)|}{\beta!} \ . \quad (5.12)$$

Since $L > \sigma_p - s + \alpha_1 = -s + \alpha_1$ and $\lambda \in b_{pq}^{s,mloc}(\boldsymbol{w}) \hookrightarrow b_{p\infty}^{s,mloc}(\boldsymbol{w})$, the convergence of (5.8) in $\mathcal{S}'(\mathbb{R}^n)$ follows.
If $p < 1$, we get analogously by choosing $\kappa = \alpha$, $M > L + n + 2\alpha$ and using Hölder's inequality and the weight property $w_\nu(y) \leq C w_\nu(2^{-\nu}m)(1 + 2^\nu|y - 2^{-\nu}m|)^\alpha$

$$\left| \int_{\mathbb{R}^n} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\mu_{\nu m}(y)\varphi(y)dy \right|^p$$
$$\leq c 2^{-\nu(L+s-\alpha_1)} 2^{\nu s} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^p w_\nu^p(2^{-\nu}m) \int_{\mathbb{R}^n} (1 + 2^\nu|y - 2^{-\nu}m|)^{L+2\alpha-M} dy \|\varphi\|_{\alpha,L}$$
$$\leq c' 2^{-\nu(L+s+n-\frac{n}{p}-\alpha_1)} 2^{\nu(s-\frac{n}{p})} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^p w_\nu^p(2^{-\nu}m) \|\varphi\|_{\alpha,L} \ .$$

Finally, using $\ell_p \hookrightarrow \ell_1$, $L > \sigma_p - s + \alpha_1 = n\left(\frac{1}{p} - 1\right) - s + \alpha_1$ and $\lambda \in b_{pq}^{s,mloc}(\boldsymbol{w}) \hookrightarrow b_{p\infty}^{s,mloc}(\boldsymbol{w})$ we get the $\mathcal{S}'(\mathbb{R}^n)$ convergence of (5.8). □

Remark 5.8: (a) The number M has to fulfil $M > L + 2\alpha + 2n$.

(b) By Remark 5.6 we also get the convergence of the atomic decomposition for all $[K, L]$-atoms with $L > \sigma_p - s + \alpha_1$ and $K \in \mathbb{N}_0$ arbitrary.

Before coming to the atomic decomposition theorem, we need a partition of unity of Calderon type.

Lemma 5.9: *Let $\{\varphi_j\}_{j\in\mathbb{N}_0} \in \Phi(\mathbb{R}^n)$ be a resolution of unity and let $M \in \mathbb{N}$. Then there exist functions $\theta_0, \theta \in \mathcal{S}(\mathbb{R}^n)$ with:*

$$\operatorname{supp}\theta_0, \operatorname{supp}\theta \subseteq \{x \in \mathbb{R}^n : |x| \leq 1\} \ , \quad (5.13)$$
$$|\hat{\theta}_0(\xi)| \geq c_0 > 0 \quad \textit{for} \quad |\xi| \leq 2 \ , \quad (5.14)$$
$$|\hat{\theta}(\xi)| \geq c > 0 \quad \textit{for} \quad \frac{1}{2} \leq |\xi| \leq 2 \ , \quad (5.15)$$
$$\int_{\mathbb{R}^n} x^\gamma\theta(x)dx = 0 \quad \textit{for} \quad 0 \leq |\gamma| \leq M \quad (5.16)$$

and

$$\hat{\theta}_0(\xi)\hat{\psi}_0(\xi) + \sum_{j=1}^{\infty} \hat{\theta}(2^{-j}\xi)\hat{\psi}(2^{-j}\xi) = 1 \quad , \textit{ for all } \xi \in \mathbb{R}^n \ , \tag{5.17}$$

where the functions $\psi_0, \psi \in \mathcal{S}(\mathbb{R}^n)$ *are defined by*

$$\hat{\psi}_0(\xi) = \frac{\varphi_0(\xi)}{\hat{\theta}_0(\xi)} \quad \textit{and} \quad \hat{\psi}(\xi) = \frac{\varphi_1(2\xi)}{\hat{\theta}(\xi)} \ . \tag{5.18}$$

Proof: Let, as in [FrJa85, Theorem 2.6], $\Theta \in \mathcal{S}(\mathbb{R}^n)$ be a radial function with $\operatorname{supp}\Theta \subseteq \{x \in \mathbb{R}^n : |x| \leq 1\}$ and $\hat{\Theta}(0) = 1$. Then, we have for some $1 > \varepsilon > 0$:

$$\hat{\Theta}(\xi) \geq \frac{1}{2} \quad \text{for all } |\xi| \leq 2\varepsilon.$$

Now, defining

$$\theta(x) = \varepsilon^{-n}(-\Delta)^M \Theta(x/\varepsilon) \ , \tag{5.19}$$

we get that θ satisfies (5.13)-(5.17). Since $\hat{\theta}(\xi) \geq c > 0$ for $\frac{1}{2} \leq |\xi| \leq 2$, we get that $\hat{\theta}(\xi) > 0$ for all $\xi \in \operatorname{supp}\varphi_1(2\cdot)$. Therefore, ψ in (5.18) is well defined. Furthermore, we have

$$\sum_{j=1}^{\infty} \hat{\theta}(2^{-j}\xi)\hat{\psi}(2^{-j}\xi) = 1 \quad \text{for all } \xi \in \mathbb{R}^n \setminus \operatorname{supp}\varphi_0.$$

In a similar way one finds $\theta_0 \in \mathcal{S}(\mathbb{R}^n)$ with $\hat{\theta}_0(\xi) \geq c_0 > 0$ for all $\xi \in \operatorname{supp}\varphi_0$:
Let $\Theta_0 \in \mathcal{S}(\mathbb{R}^n)$ with $\operatorname{supp}\Theta_0 \subseteq \{x \in \mathbb{R}^n : |x| \leq 1\}$ and for some $\delta \in (0,1)$

$$\hat{\Theta}_0(\xi) \geq \frac{1}{2} \quad \text{for all } |\xi| \leq 2\delta \ .$$

Then $\theta_0(x) = \delta^{-n}\Theta_0(x/\delta)$ satisfies all conditions (5.13)-(5.17) and the lemma is proved. □

We have already seen that the sum in (5.8) converges in $\mathcal{S}'(\mathbb{R}^n)$ under the conditions of Lemma 5.7. Now we come to the atomic decomposition theorem which explains the limit element.

Theorem 5.10: *Let* $\boldsymbol{w} = \{w_k\}_{k\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $s \in \mathbb{R}$ *and let* $0 < p, q \leq \infty$. *Furthermore, let* $K, L \in \mathbb{N}_0$ *with*

$$K > s + \alpha_2 \qquad \textit{and} \qquad L > \sigma_p - s + \alpha_1 \ . \tag{5.20}$$

For every $f \in B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ *there exists* $\lambda \in b^{s,mloc}_{pq}(\boldsymbol{w})$ *and* $[K, L]$*-atoms* $\{a_{\nu m}\}$ *centered at* $Q_{\nu m}$ *such that there exists a representation*

$$f = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m} \quad \textit{converging in } \mathcal{S}'(\mathbb{R}^n), \tag{5.21}$$

with

$$\left\| \lambda | \, b^{s,mloc}_{pq}(\boldsymbol{w}) \right\| \le c \left\| f | \, B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right\| \ ,$$

where the constant c is universal for all $f \in B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$.

Proof: It is not a special assumption that the sum (5.21) has to converge. Since $[K, L]$-atoms are $[K, L, M]$-molecules for every $M > 0$ the convergence in $\mathcal{S}'(\mathbb{R}^n)$ has already been proven in Lemma 5.7.
The proof relies on the method used in the proof of [FJW91, Theorem 5.11] and the idea of using the local means is from [FaLeo06].
Therefore, we use Lemma 5.9 with $M = L-1$, the functions $\theta_0, \theta \in \mathcal{S}(\mathbb{R}^n)$ with the properties (5.13)-(5.17) and the functions $\psi_0, \psi \in \mathcal{S}(\mathbb{R}^n)$ with (5.18). Let $f \in B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$, then we get from the lemma

$$f = f * \theta_0 * \psi_0 + \sum_{\nu=1}^{\infty} 2^{\nu n} \theta(2^\nu \cdot) * \psi_\nu * f \ ,$$

where $\psi_\nu(\cdot) = 2^{\nu n}\psi(2^\nu \cdot)$. Now, using the cubes $Q_{\nu m}$ we derive

$$f(x) = \sum_{m \in \mathbb{Z}^n} \int\limits_{Q_{0m}} \theta_0(x-y)(\psi_0 * f)(y) dy + \sum_{\nu=1}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{\nu n} \int\limits_{Q_{\nu m}} \theta(2^\nu(x-y))(\psi_\nu * f)(y) dy \ . \tag{5.22}$$

We define for every $\nu \in \mathbb{N}$ and all $m \in \mathbb{Z}^n$

$$\lambda_{\nu m} = C_\theta \sup_{y \in Q_{\nu m}} |(\psi_\nu * f)(y)| \ , \tag{5.23}$$

where $C_\theta = \max\{\sup_{|x| \le 1} |D^\beta \theta(x)| : |\beta| \le K\}$. If $\lambda_{\nu m} \neq 0$, then we define

$$a_{\nu m}(x) = \frac{1}{\lambda_{\nu m}} 2^{\nu n} \int\limits_{Q_{\nu m}} \theta(2^\nu(x-y))(\psi_\nu * f)(y) dy \ , \tag{5.24}$$

otherwise we set $a_{\nu m}(x) = 0$. The a_{0m} atoms and λ_{0m} are defined similar using θ_0 and ψ_0. Clearly, (5.21) is satisfied with these atoms and coefficients and the properties of θ_0, ψ_0, θ and ψ_ν ensure that $a_{\nu m}$ are $[K, L]$-atoms. It remains to prove that there exists a constant c such that $\left\| \lambda | \, b^{s,mloc}_{pq}(\boldsymbol{w}) \right\| \le c \left\| f | \, B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right\|$.
We have for fixed $\nu \in \mathbb{N}_0$ and $a > \frac{n}{p} + \alpha$

$$\begin{aligned}
\sum_{m \in \mathbb{Z}^n} w_\nu(x) \lambda_{\nu m} \chi_{\nu m}(x) &\le c \sum_{m \in \mathbb{Z}^n} w_\nu(x) \sup_{y \in Q_{\nu m}} |(\psi_\nu * f)(y)| \chi_{\nu m}(x) \\
&\le c' w_\nu(x) \sup_{|z| \le \gamma 2^{-\nu}} \frac{|(\psi_\nu * f)(x-z)|}{1 + |2^\nu z|^a} (1 + |2^\nu z|^a) \\
&\le c'' w_\nu(x) (\psi_\nu^* f)_a(x) \ ,
\end{aligned}$$

since $|x-y| \le c2^{-\nu}$ for $x, y \in Q_{\nu m}$ and $\sum_{m\in\mathbb{Z}^n} \chi_{\nu m}(x) = 1$. Here, $(\psi_\nu^* f)_a$ denotes the Peetre maximal operator, defined in (3.3). Therefore, we have using Remark 5.2

$$\left\| \lambda | \, b_{pq}^{s,mloc}(\boldsymbol{w}) \right\| \le c \left(\sum_{\nu=0}^{\infty} 2^{\nu sq} \left\| w_\nu(x) (\psi_\nu^* f)_a(x) | \, L_p(\mathbb{R}^n) \right\|^q \right)^{1/q} . \tag{5.25}$$

Since $\psi_0 \in \mathcal{S}(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$ are two kernels which satisfy the moment conditions (3.35) and the Tauberian conditions (3.36) and (3.37), we can use Theorem 3.8 with $a > \frac{n}{p} + \alpha$ and derive from (5.25)

$$\left\| \lambda | \, b_{pq}^{s,mloc}(\boldsymbol{w}) \right\| \le c \left\| f | \, B_{pq}^{s,mloc}(\mathbb{R}^n, \boldsymbol{w}) \right\| .$$

□

To prove the other direction of the atomic decomposition, we take the more general molecules in the above sense. Therefore, we need three technical lemmas as used in [FrJa85]. We want to estimate the size of $\varphi_j^\vee * \mu_{\nu m}$, where $\{\varphi_j\} \in \Phi(\mathbb{R}^n)$ and $\mu_{\nu m}$ is a $[K, L, M]$-molecule concentrated in $Q_{\nu m}$. We use the resolution of unity introduced in (3.1), especially we use the function φ with $\varphi_j = \varphi(2^{-j}\cdot)$. We have for $j \ge 1$

$$(\varphi_j^\vee * \mu_{\nu m})(x) = (\varphi^\vee * \mu)(2^j(x - 2^{-\nu}m)) \tag{5.26}$$

where

$$\mu(x) = \mu_{\nu m}(2^{-j}x + 2^{-\nu}m) . \tag{5.27}$$

From Definition 5.5 we get

$$|D^\beta \mu(x)| \le c 2^{(\nu-j)|\beta|}(1 + 2^{\nu-j}|x|)^{-M} \quad \text{for } |\beta| \le K \text{ and} \tag{5.28}$$

$$\int_{\mathbb{R}^n} x^\beta \mu(x) dx = 0 \quad \text{for } |\beta| < L \text{ and } \nu \ge 1. \tag{5.29}$$

First, we prove some technical lemmas for this special function μ and later on we combine them to get a corollary for the $\mu_{\nu m}$.

Lemma 5.11: *Let $L \in \mathbb{N}_0$, $M > L + n$ and let $\lambda \ge 0$. Further, let $\mu : \mathbb{R}^n \to \mathbb{C}$ be a continuous function with*

$$|\mu(x)| \le c(1 + 2^\lambda |x|)^{-M} \quad \textit{and} \tag{5.30}$$

$$\int_{\mathbb{R}^n} x^\beta \mu(x) dx = 0 \quad \textit{for all } |\beta| < L. \tag{5.31}$$

Then for all $g \in \mathcal{S}(\mathbb{R}^n)$

$$|(g * \mu)(x)| \le c 2^{-\lambda(L+n)}(1 + |x|)^{L+n-M} . \tag{5.32}$$

Proof: We use Taylor expansion of g and derive from the moment conditions on μ

$$(g*\mu)(x) = \int\limits_{\mathbb{R}^n} \mu(x-y) \left(g(y) - \sum_{|\beta|<L} \frac{D^\beta g(x)}{\beta!}(x-y)^\beta \right) dy \ .$$

Therefore

$$\begin{aligned}|(g*\mu)(x)| &\le c \int\limits_{\mathbb{R}^n} |\mu(x-y)||x-y|^L \Psi(x,y)dy \\ &\le c \int\limits_{|x-y|\le\frac{|x|}{2}} |\mu(x-y)||x-y|^L \Psi(x,y)dy + c \int\limits_{|x-y|\ge\frac{|x|}{2}} |\mu(x-y)||x-y|^L \Psi(x,y)dy \\ &=: I + II \ ,\end{aligned}$$

where

$$\Psi(x,y) = \sup_{|\beta|=L} \sup_{0<\theta<1} \left| D^\beta g(x - \theta(x-y)) \right| \ .$$

Since $g \in \mathcal{S}(\mathbb{R}^n)$ we find for every $\kappa > 0$ a constant c_κ such that

$$\Psi(x,y) \le c_\kappa \sup_{0<\theta<1} (1 + |x - \theta(x-y)|)^{-\kappa} \ .$$

Now, using this and (5.30) we achieve

$$\begin{aligned} I &\le c \int\limits_{|x-y|\le\frac{|x|}{2}} (1+2^\lambda|x-y|)^{-M} |2^\lambda(x-y)|^L 2^{-\lambda L} \left(1 + \left| |x| - \frac{|x|}{2} \right| \right)^{-\kappa} dy \\ &\le c' 2^{-\lambda(L+n)} (1+|x|)^{-\kappa} \ , \qquad \text{for } M > L+n. \end{aligned} \tag{5.33}$$

The second term can be estimated in using again (5.30) and $\Psi(x,y) \le c$

$$\begin{aligned} II &\le c \int\limits_{|x-y|\ge\frac{|x|}{2}} (1+2^\lambda|x-y|)^{-M} |2^\lambda(x-y)|^L 2^{-\lambda L} dy \\ &\le c' 2^{-\lambda(L+n)} \int\limits_{|z|\ge 2^\lambda\frac{|x|}{2}} (1+|z|)^{-M+L} dz \\ &\le c'' 2^{-\lambda(L+n)} (1+|x|)^{-M+L+n} \ . \end{aligned} \tag{5.34}$$

Finally, the lemma follows from (5.34) and (5.33) with $\kappa = -M + L + n$. $\square$

Lemma 5.12: *Let $K \in \mathbb{N}_0$, $M > 0$ and let $\lambda \le 0$. Further, let $\mu \in C^K(\mathbb{R}^n)$ with*

$$|D^\beta \mu(x)| \le c 2^{\lambda K} (1+2^\lambda|x|)^{-M} \ , \quad \textit{for } |\beta| = K. \tag{5.35}$$

Then for all $g_0 \in \mathcal{S}(\mathbb{R}^n)$ with $\operatorname{supp} \hat{g}_0 \subseteq \{\xi \in \mathbb{R}^n : 1/2 \le |\xi| \le 2\}$ we have

$$|(g_0 * \mu)(x)| \le c 2^{-\lambda K} (1+2^\lambda|x|)^{-M} \ . \tag{5.36}$$

Proof: In contrast to the previous Lemma 5.11 we use now the moment conditions of g_0 (5.36) and the decay properties on the K-th derivatives of μ (5.35). In analogy to the previous proof we derive by Taylor expansion of μ with order $K-1$

$$|(g_0 * \mu)(x)| \le c \int\limits_{|x-y|\le\frac{|x|}{2}} |g_0(x-y)||x-y|^K \Psi(x,y)dy + c \int\limits_{|x-y|\ge\frac{|x|}{2}} |g_0(x-y)||x-y|^K \Psi(x,y)dy$$
$$=: I + II \ . \tag{5.37}$$

Here, the remainder term $\Psi(x,y)$ can be estimated by (5.35)

$$\begin{aligned}\Psi(x,y) &= \sup_{|\beta|=K} \sup_{0<\theta<1} |D^\beta \mu(x-\theta(y-x))| \\ &\le c2^{\lambda K} \sup_{0<\theta<1} (1+2^\lambda|x-\theta(y-x)|)^{-M} \ .\end{aligned}$$

For $g_0 \in \mathcal{S}(\mathbb{R}^n)$ we get for every $\kappa > 0$

$$|g_0(x)| \le c_\kappa (1+|x|)^\kappa \ .$$

If we choose κ large enough the first term in (5.37) gives

$$\begin{aligned}I &\le c2^{\lambda K} \int\limits_{|x-y|\le\frac{|x|}{2}} (1+|x-y|)^{K-\kappa} dy (1+2^\lambda|x|)^{-M} \\ &\le c' 2^{\lambda K}(1+2^\lambda|x|)^{-M} \ .\end{aligned}$$

For the second term in (5.37) we use $|\Psi(x,y)| \le c2^{\lambda K}$ and derive for $\kappa = M-K-n$ by direct integration

$$\begin{aligned}II &\le c2^{\lambda K} \int\limits_{|x-y|\ge\frac{|x|}{2}} (1+|x-y|)^{K-\kappa} dy \\ &\le c' 2^{\lambda K}(1+|x|)^{K+n-\kappa} \le c' 2^{\lambda K}(1+|x|)^{-M} \ .\end{aligned}$$

Finally, since $\lambda \le 0$ we get now the result we aimed at. □

Now, we come back to the situation where we want to estimate $|\varphi_j * \mu_{\nu m}(x)|$, where $\{\varphi_j\} \in \Phi(\mathbb{R}^n)$ and $\mu_{\nu m}$ is a $[K,L,M]$-molecule concentrated at $Q_{\nu m}$. Therefore, we combine the previous lemmas.

Corollary 5.13: *Let $\{\varphi_j\}_{j\in\mathbb{N}_0} \in \Phi(\mathbb{R}^n)$ be a resolution of unity and $\{\mu_{\nu m}\}_{\nu\in\mathbb{N}_0, m\in\mathbb{Z}^n}$ are $[K,L,M]$-molecules. Then we have for all $x \in \mathbb{R}^n$*

$$|(\varphi_j^\vee * \mu_{\nu m})(x)| \le c2^{-(\nu-j)(L+n)}(1+2^j|x-2^{-\nu}m|)^{L+n-M} \ , \quad \text{for } j \le \nu \tag{5.38}$$

and

$$|(\varphi_j^\vee * \mu_{\nu m})(x)| \le c2^{-(j-\nu)K}(1+2^j|x-2^{-\nu}m|)^{-M} \ , \quad \text{for } j \ge \nu. \tag{5.39}$$

Proof: We use $(\varphi_j^\vee * \mu_{\nu m})(x) = (\mu * \varphi^\vee)(2^j(x - 2^{-\nu}m))$. Then we can use for the first case Lemma 5.11 and for the second case we use Lemma 5.12. Only the case $\nu = j = 0$ has to be treated separately. But this is an easy modification of the lemmas. $\square$

Remark 5.14: The corollary is analogous to Lemma 3.3 in [FrJa85].

A last lemma is needed before we can formulate the decomposition theorem by molecules.

Lemma 5.15: *Let $1 \le p \le \infty$, $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ and let*

$$F(x) = \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} f_{\nu m} \ ,$$

where $\lambda_{\nu m} \in \mathbb{C}$ and $f_{\nu m}(x) = (1 + 2^j|x - 2^{-\nu}m|)^{-R}$ for $R > \alpha + n$.
Then we have for $j \le \nu$

$$\| F |\, L_p(\mathbb{R}^n, w_j)\| \le c 2^{-\nu\frac{n}{p}} 2^{(\nu-j)(\alpha_1+n)} \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^p w_\nu^p(2^{-\nu}m) \right)^{1/p}$$

and for $j \ge \nu$

$$\| F |\, L_p(\mathbb{R}^n, w_j)\| \le c 2^{-\nu\frac{n}{p}} 2^{(j-\nu)\alpha_2} \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^p w_\nu^p(2^{-\nu}m) \right)^{1/p} .$$

Proof: First Step: We treat the case $j \le \nu$. Therefore, we decompose $\mathbb{R}^n$ into cubes $Q_{\nu l}$ and get

$$\begin{aligned} \| F |\, L_p(\mathbb{R}^n, w_j)\|^p &= \int_{\mathbb{R}^n} w_j^p(x) \left| \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} f_{\nu m}(x) \right|^p dx \\ &\le c \sum_{l\in\mathbb{Z}^n} \int_{Q_{\nu l}} \left| \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} w_j(x) (1 + 2^j|x - 2^{-\nu}m|)^{-R} \right|^p dx \ . \end{aligned}$$

Now, we use

$$w_j(x) \le \mathrm{C} 2^{\alpha_1(\nu-j)} w_\nu(2^{-\nu}m)(1 + 2^j|x - 2^{-\nu}m|)^{\alpha}$$

and that $x \in Q_{\nu l}$ (that means $0 \le |x - 2^{-\nu}l| \le c2^{-\nu}$). Using this we derive

$$\begin{aligned} \| F |\, L_p(w_j)\|^p &\le c 2^{\alpha_1 p(\nu-j)} \sum_{l\in\mathbb{Z}^n} \int_{Q_{\nu l}} \left| \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} w_\nu(2^{-\nu}m)(1 + 2^{j-\nu}|l - m|)^{-R+\alpha} \right|^p dx \\ &\le c 2^{\alpha_1 p(\nu-j)} 2^{-\nu n} \sum_{l\in\mathbb{Z}^n} \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}| w_\nu(2^{-\nu}m)(1 + 2^{j-\nu}|l - m|)^{-R+\alpha} \right)^p \end{aligned}$$

and Young's inequality gives us

$$\leq c2^{\alpha_1 p(\nu-j)}2^{-\nu n}\left(\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|^p w_\nu^p(2^{-\nu}m)\right)\left(\sum_{l\in\mathbb{Z}^n}(1+2^{j-\nu}|l|)^{-R+\alpha}\right)^p . \tag{5.40}$$

Finally, we have to estimate the last sum in (5.40). Therefore, we split the sum and obtain

$$\begin{aligned}\sum_{l\in\mathbb{Z}^n}(1+2^{j-\nu}|l|)^{-R+\alpha} &= \sum_{|l|\leq 2^{\nu-j}}(1+2^{j-\nu}|l|)^{-R+\alpha}+\sum_{|l|>2^{\nu-j}}(1+2^{j-\nu}|l|)^{-R+\alpha}\\ &\leq c2^{(\nu-j)n}+\int\limits_{|y|>2^{\nu-j}}(1+2^{j-\nu}|y|)^{-R+\alpha}dy\\ &\leq c'2^{(\nu-j)n} , \qquad \text{for } R>\alpha+n.\end{aligned}$$

Putting this into (5.40) and taking the $1/p$ power, the first part of the lemma is proved.
Second Step: Now, we have $j\geq\nu$ and therefore we can use

$$f_{\nu m}(x)=(1+2^j|x-2^{-\nu}m|)^{-R}\leq(1+2^\nu|x-2^{-\nu}m|)^{-R}$$

and

$$w_j(x)\leq \mathrm{C}2^{\alpha_2(j-\nu)}w_\nu(2^{-\nu}m)(1+2^\nu|x-2^{-\nu}m|)^\alpha .$$

The same splitting of the integral in dyadic cubes as in the first step together with the above inequalities are proving the second case. □

Remark 5.16: This lemma is a generalization of Lemma 3.4 in [FrJa85] for $w_j\equiv 1$.

Finally, we can state the other direction of the decomposition theorem by molecules.

Theorem 5.17: *Let $\boldsymbol{w}\in\mathcal{W}^\alpha_{\alpha_1,\alpha_2}$, $s\in\mathbb{R}$ and let $0<p,q\leq\infty$. Further, let $K,L\in\mathbb{N}_0$ with*

$$K>s+\alpha_2 , \qquad L>\sigma_p-s+\alpha_1 \tag{5.41}$$

and $M>0$ large enough ($M>L+n(\max(1,1/p)+1)+2\alpha$). If $\{\mu_{\nu m}\}$ are $[K,L,M]$-molecules and $\lambda=\{\lambda_{\nu m}\}\in b^{s,mloc}_{pq}(\boldsymbol{w})$, then

$$f=\sum_{\nu=0}^\infty\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\mu_{\nu m} , \qquad \textit{convergence in } \mathcal{S}'(\mathbb{R}^n), \tag{5.42}$$

is an element of $B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})$ and

$$\left\|f|\,B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\right\|\leq c\left\|\lambda|\,b^{s,mloc}_{pq}(\boldsymbol{w})\right\| ,$$

where the constant $c>0$ is independent of $\lambda\in b^{s,mloc}_{pq}(\boldsymbol{w})$.

Proof: We have the representation of $f \in \mathcal{S}'(\mathbb{R}^n)$ by (5.42) and we know by Lemma 5.7 that this representation converges. Now, we estimate the norm of f

$$\begin{aligned}
&\big\| f \big| B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})\big\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| (\varphi_j \hat{f})^\vee w_j \right| L_p(\mathbb{R}^n)\right\|^q \right)^{1/q} \\
&\le c \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} (\varphi_j^\vee * \mu_{\nu m}) w_j \right| L_p(\mathbb{R}^n)\right\|^q \right)^{1/q} \\
&\le c \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| \sum_{\nu=0}^{j} \sum_{m\in\mathbb{Z}^n} \ldots \right| L_p(\mathbb{R}^n)\right\|^q \right)^{1/q} + c \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| \sum_{\nu=j+1}^{\infty} \sum_{m\in\mathbb{Z}^n} \ldots \right| L_p(\mathbb{R}^n)\right\|^q \right)^{1/q} .
\end{aligned} \tag{5.43}$$

Let $p \ge 1$. We estimate the first L_p Norm in (5.43). From Corollary 5.13 we have $|(\varphi_j^\vee * \mu_{\nu m})(x)| \le c 2^{-(j-\nu)K}(1 + 2^j|x - 2^{-\nu}m|)^{-M}$ and we get by Lemma 5.15 with $M > n + \alpha$

$$\begin{aligned}
I &:= \left\| \sum_{\nu=0}^{j} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} (\varphi_j^\vee * \mu_{\nu m}) w_j \right| L_p(\mathbb{R}^n)\right\| \\
&\le c \sum_{\nu=0}^{j} 2^{-\nu \frac{n}{p}} 2^{-(j-\nu)(K-\alpha_2)} \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^p w_\nu^p(2^{-\nu}m)\right)^{1/p} .
\end{aligned} \tag{5.44}$$

For the second term in (5.43) we use also Corollary 5.13, hence $|(\varphi_j^\vee * \mu_{\nu m})(x)| \le c 2^{-(\nu-j)(L+n)}(1 + 2^j|x - 2^{-\nu}m|)^{-M+L+n}$, and Lemma 5.15 with $M > L + 2n + \alpha$ and get

$$\begin{aligned}
II &:= \left\| \sum_{\nu=j+1}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} (\varphi_j^\vee * \mu_{\nu m}) w_j \right| L_p(\mathbb{R}^n)\right\| \\
&\le c \sum_{\nu=j+1}^{\infty} 2^{-\nu \frac{n}{p}} 2^{-(\nu-j)(L-\alpha_1)} \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^p w_\nu^p(2^{-\nu}m)\right)^{1/p} .
\end{aligned} \tag{5.45}$$

Now, we consider the case $0 < p < 1$. We use the embedding $\ell_p \hookrightarrow \ell_1$ and obtain for the first term in (5.43)

$$\begin{aligned}
I &\le \left(\int_{\mathbb{R}^n} \left(\sum_{\nu=0}^{j} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}| |(\varphi_j^\vee * \mu_{\nu m})(x)| w_j(x) \right)^p dx \right)^{1/p} \\
&\le c \left(\sum_{\nu=0}^{j} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^p \int_{\mathbb{R}^n} |(\varphi_j^\vee * \mu_{\nu m})(x)|^p w_j^p(x) dx \right)^{1/p}
\end{aligned}$$

With the help of Corollary 5.13 we can estimate $|(\varphi_j^\vee * \mu_{\nu m})(x)|$ in the usual way and using the properties of the weight sequence we estimate

$$\int_{\mathbb{R}^n} |(\varphi_j^\vee * \mu_{\nu m})(x)|^p w_j^p(x) dx \leq c 2^{-(j-\nu)(K-\alpha_2)p} \int_{\mathbb{R}^n} (1 + 2^\nu |x - 2^{-\nu} m|)^{(-M+\alpha)p} w_\nu^p(2^{-\nu} m) dx$$
$$\leq c' 2^{-(j-\nu)(K-\alpha_2)p} 2^{-\nu n} w_\nu^p(2^{-\nu} m) \ , \qquad \text{for } M > \frac{n}{p} + \alpha.$$

So, we get for the first L_p norm

$$I \leq c \left(\sum_{\nu=0}^{j} 2^{-\nu n} 2^{-(j-\nu)(K-\alpha_2)p} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p w_\nu^p(2^{-\nu} m) \right)^{1/p} . \tag{5.46}$$

By a similar calculation we obtain the second estimate ($M > \frac{n}{p} + L + n + \alpha$)

$$II \leq c \left(\sum_{\nu=j+1}^{\infty} 2^{-\nu n} 2^{-(\nu-j)(L-\sigma_p-\alpha_1)p} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p w_\nu^p(2^{-\nu} m) \right)^{1/p} . \tag{5.47}$$

We denote $\tilde{p} := \min(1, p)$ and $t := \min(1, p, q)$. We can rewrite our results for the first term as

$$I \leq c \left(\sum_{\nu=0}^{j} 2^{-\nu \frac{n}{p} \tilde{p}} 2^{-(j-\nu)(K-\alpha_2)\tilde{p}} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p w_\nu^p(2^{-\nu} m) \right)^{\tilde{p}/p} \right)^{1/\tilde{p}} ,$$

for all $0 < p, q \leq \infty$. Finally, we conclude with that notation and with $\ell_{t/\tilde{p}} \hookrightarrow \ell_1$

$$\left\| 2^{js} I \right| \ell_q \right\|^t \leq c \left\| \sum_{\nu=0}^{j} 2^{\nu(s-\frac{n}{p})t} 2^{-(j-\nu)(K-s-\alpha_2)t} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p w_\nu^p(2^{-\nu} m) \right)^{t/p} \right| \ell_{q/t} \right\| ,$$

and Young's inequality gives us with $\zeta := K - s - \alpha_2 > 0$

$$\leq c' \left(\sum_{\nu=0}^{\infty} 2^{\nu(s-\frac{n}{p})q} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p w_\nu^p(2^{-\nu} m) \right)^{q/p} \right)^{t/q} \left(\sum_{j=0}^{\infty} 2^{-j\zeta t} \right)$$
$$= c'' \left\| \lambda \right| b_{pq}^{s,mloc}(\boldsymbol{w}) \right\|^t .$$

With the same notation a similar estimation can be done for $\left\| 2^{js} II \right| \ell_q \right\|^t$. Here one has to use $\zeta := L - \sigma_p + s - \alpha_1 > 0$ and this finishes the proof. □

For every $M > 0$ every $[K, L]$ atom is a $[K, L, M]$ molecule. So we get an easy corollary for the atomic decomposition.

Corollary 5.18: *Let $\boldsymbol{w} = \{w_k\}_{k \in \mathbb{N}_0} \in \mathcal{W}_{\alpha_1,\alpha_2}^{\alpha}$ and let $0 < p, q \leq \infty$, $s \in \mathbb{R}$. Furthermore, let $K, L \in \mathbb{N}_0$ with*

$$K > s + \alpha_2 \qquad \text{and} \qquad L > \sigma_p - s + \alpha_1 \ .$$

(i) *If* $\lambda \in b_{pq}^{s,mloc}(\boldsymbol{w})$ *and* $\{a_{\nu m}\}_{\nu\in\mathbb{N}_0, m\in\mathbb{Z}^n}$ *are* $[K,L]$*-atoms centered at* $Q_{\nu m}$*, then*

$$f = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m} \tag{5.48}$$

belongs to the space $B_{pq}^{s,mloc}(\mathbb{R}^n, \boldsymbol{w})$ *and there exists a constant* $c > 0$ *with*

$$\left\| f | \, B_{pq}^{s,mloc}(\mathbb{R}^n, \boldsymbol{w}) \right\| \leq c \left\| \lambda | \, b_{pq}^{s,mloc}(\boldsymbol{w}) \right\| \ .$$

The constant c *is universal for all* λ *and* $a_{\nu m}$*.*

(ii) *For every* $f \in B_{pq}^{s,mloc}(\mathbb{R}^n, \boldsymbol{w})$ *there exists* $\lambda \in b_{pq}^{s,mloc}(\boldsymbol{w})$ *and* $[K,L]$*-atoms centered at* $Q_{\nu m}$ *such that there exists a representation* (5.48)*, converging in* $\mathcal{S}'(\mathbb{R}^n)$*, with*

$$\left\| \lambda | \, b_{pq}^{s,mloc}(\boldsymbol{w}) \right\| \leq c \left\| f | \, B_{pq}^{s,mloc}(\mathbb{R}^n, \boldsymbol{w}) \right\| \ ,$$

where the constant c *is universal for all* $f \in B_{pq}^{s,mloc}(\mathbb{R}^n, \boldsymbol{w})$*.*

5.3 Wavelet decomposition

In this section we describe the characterization of $B_{pq}^{s,mloc}(\mathbb{R}^n, \boldsymbol{w})$ by a decomposition in wavelets. We follow closely the ideas in [Tri04], [Tri08] and [Kyr03].

5.3.1 Preliminaries

First of all, we recall some results from wavelet theory.

Theorem 5.19: *(i) There is a real scaling function* $\psi_F \in \mathcal{S}(\mathbb{R})$ *and a real associated wavelet* $\psi_M \in \mathcal{S}(\mathbb{R})$ *such that their Fourier transforms have compact supports,* $\widehat{\psi_F}(0) = (2\pi)^{-1/2}$ *and*

$$\operatorname{supp} \widehat{\psi_M} \subseteq \left[-\frac{8}{3}\pi, -\frac{2}{3}\pi\right] \cup \left[\frac{2}{3}\pi, \frac{8}{3}\pi\right] \ . \tag{5.49}$$

(ii) *For any* $k \in \mathbb{N}$ *there is a real compactly supported scaling function* $\psi_F \in C^k(\mathbb{R})$ *and a real compactly supported associated wavelet* $\psi_M \in C^k(\mathbb{R})$ *such that* $\widehat{\psi_F}(0) = (2\pi)^{-1/2}$ *and*

$$\int_{\mathbb{R}} x^l \psi_M(x) dx = 0 \qquad \text{for all } l \in \{0, \ldots, k-1\} \ . \tag{5.50}$$

In both cases we have that $\{\psi_{\nu m} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}\}$ *is an orthonormal basis in* $L_2(\mathbb{R})$*, where*

$$\psi_{\nu m}(t) := \begin{cases} \psi_F(t-m) & , \text{ if } \nu = 0, m \in \mathbb{Z} \\ 2^{\frac{\nu-1}{2}} \psi_M(2^{\nu-1}t - m) & , \text{ if } \nu \in \mathbb{N}, m \in \mathbb{Z} \end{cases} \tag{5.51}$$

and the functions ψ_M, ψ_F *are according to (i) or (ii).*

Remark 5.20: The wavelets in the first part are called Meyer wavelets. They do not have a compact support but they are fast decaying functions ($\psi_F, \psi_M \in \mathcal{S}(\mathbb{R})$) and ψ_M has infinitely many moment conditions. The wavelets from the second part are called Daubechies Wavelets. Here the functions ψ_M, ψ_F do have compact support, but they only have limited smoothness. Both types of wavelets are well described in [Woj97], Chapters 3 and 4.

Now, we generalize this to the n-dimensional case by a tensor product procedure. We take over the notation from [Tri06, Section 4.2.1] with $l = 0$. Let ψ_M, ψ_F be the Meyer or Daubechies wavelets described above. Now, we define

$$G^0 = \{F, M\}^n \qquad \text{and} \qquad G^\nu = \{F, M\}^{n*} \quad \text{if } \nu \geq 1 \ , \tag{5.52}$$

where the * indicates that at least one G_i of $G = (G_1, \dots, G_n) \in \{F, M\}^{n*}$ must be an M. It is clear from the definition that the cardinal number of $\{F, M\}^{n*}$ is $2^n - 1$. Let for $x \in \mathbb{R}$

$$\Psi^\nu_{Gm}(x) = 2^{\nu \frac{n}{2}} \prod_{r=1}^{n} \psi_{G_r}(2^\nu x_r - m_r) \ , \tag{5.53}$$

where $\nu \in \mathbb{N}_0$, $G \in G^\nu$ and $m \in \mathbb{Z}^n$. Then it follows by Theorem 5.19 that $\{\Psi^\nu_{Gm} : \nu \in \mathbb{N}_0, G \in G^\nu, m \in \mathbb{Z}^n\}$ is an orthonormal basis in $L_2(\mathbb{R}^n)$. Finally, we have to adjust the sequence spaces $b^{s,mloc}_{pq}(\boldsymbol{w})$ to our situation.

Definition 5.21: *Let $\boldsymbol{w} \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$, $s \in \mathbb{R}$ and $0 < p, q \leq \infty$. Then*

$$\tilde{b}^{s,mloc}_{pq}(\boldsymbol{w}) := \left\{ \lambda = \{\lambda^\nu_{Gm}\} \subset \mathbb{C} : \left\| \lambda | \, \tilde{b}^{s,mloc}_{pq}(\boldsymbol{w}) \right\| < \infty \right\} \quad \textit{where} \tag{5.54}$$

$$\left\| \lambda | \, \tilde{b}^{s,mloc}_{pq}(\boldsymbol{w}) \right\| = \left(\sum_{\nu=0}^{\infty} 2^{\nu(s-\frac{n}{p})q} \sum_{G \in G^\nu} \left(\sum_{m \in \mathbb{Z}^n} |\lambda^\nu_{Gm}|^p w^p_\nu(2^{-\nu}m) \right)^{q/p} \right)^{1/q} \ . \tag{5.55}$$

To get the wavelet characterization by Daubechies wavelets we use local means with kernels which only have limited smoothness and we use the molecular decomposition described in the previous section. This idea goes back to [Tri08], [Kyr03] and [FJW91]. First, we recall the local means with kernel k

$$k(t, f)(x) = t^{-n} \int_{\mathbb{R}^n} k\left(\frac{y - x}{t}\right) f(y) dy \ .$$

With $t = 2^{-j}$, $x = 2^{-j}l$ where $j \in \mathbb{N}_0$ and $l \in \mathbb{Z}^n$, one gets

$$\begin{aligned} k(2^{-j}, f)(2^{-j}l) &= 2^{jn} \int_{\mathbb{R}^n} k(2^j y - l) f(y) dy \\ &= \int_{\mathbb{R}^n} k_{jl}(y) f(y) dy \\ &= k_{jl}(f) \ . \end{aligned} \tag{5.56}$$

First, assume that the expression (5.56) makes sense, at least formally. Later on we show that (5.56) can be understand as a dual pairing. Now, the usual properties on k get shifted to the kernels k_{jl}.

Definition 5.22: *Let $k_{jl}(x) \in C^A(\mathbb{R}^n)$ with $j \in \mathbb{N}_0$ and $l \in \mathbb{Z}^n$ be functions in R^n with*

$$|D^\beta k_{jl}(x)| \le c 2^{j|\beta|+jn}(1+2^j|x-2^{-j}l|)^{-C} \ , \quad |\beta| \le A \in \mathbb{N}_0, \, C > 0 \ , \tag{5.57}$$

for all $x \in \mathbb{R}^n$, $j \in \mathbb{N}_0$, $l \in \mathbb{Z}^n$, and

$$\int_{\mathbb{R}^n} x^\beta k_{jl}(x) dx = 0 \ , \quad |\beta| < B \in \mathbb{N}_0 \ , \tag{5.58}$$

for $j \ge 1$ and $l \in \mathbb{Z}^n$.

Remark 5.23: One immediately recognizes that $\{2^{-jn}k_{jl}\}$ are $[A,B,C]$ molecules.

5.3.2 Duality

In this subsection we show that the expression

$$k(f) = \int_{\mathbb{R}^n} k(y) f(y) dy \tag{5.59}$$

makes sense as a dual pairing. Here, $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $f \in B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ and the function $k : \mathbb{R}^n \mapsto \mathbb{C}$ belongs to some weighted space of continuously differentiable functions $C^u(\mathbb{R}^n, \kappa)$,

$$C^u(\mathbb{R}^n,\kappa) := \{ f \in C^u(\mathbb{R}^n) : (1+|x|)^\kappa D^\beta f(x) \in C(\mathbb{R}^n) \quad \text{for all } |\beta| \le u \} \ ,$$

normed by

$$\| f | \, C^u(\mathbb{R}^n,\kappa) \| = \max_{|\beta| \le u} \sup_{x \in \mathbb{R}^n} (1+|x|)^\kappa |D^\beta f(x)| \ .$$

We need to introduce the dual space of $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$. By Theorem 2.32 we have that $\mathcal{S}(\mathbb{R}^n)$ is dense in $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ for $s \in \mathbb{R}$ and $\max(p,q) < \infty$. Therefore, a linear functional on $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ can be interpreted as an element of $\mathcal{S}'(\mathbb{R}^n)$. That means, $g \in \mathcal{S}'(\mathbb{R}^n)$ belongs to the dual space $(B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}))'$ of the space $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ if, and only if, there exists a positive number c such that

$$|g(\varphi)| \le c \left\| \varphi | \, B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right\| \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^n).$$

We do not want to give a full treatment of duality theory in Besov spaces, but one can modify the statements and proofs in Section 2.11 in [Tri83] and one derives at least

$$B^{\sigma_p - s, mloc}_{p'q'}(\mathbb{R}^n, \boldsymbol{w}^{-1}) \hookrightarrow \left(B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right)' \ , \tag{5.60}$$

where $\sigma_p = n(1/p-1)_+$ and

$$\frac{1}{p}+\frac{1}{p'}=1 \quad \text{if } 1\le p<\infty \quad \text{and} \quad p'=\infty \text{ if } 0<p<1. \tag{5.61}$$

The same construction (5.61) holds for q'. It is easy to see that $\boldsymbol{w^{-1}} = \{\frac{1}{w_j}\}_{j\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_2,\alpha_1}$ for $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$.

For $\max(p,q)=\infty$ we get a similar result, where the right hand side of (5.60) must be replaced by $(\mathring{B}^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w}))'$, where $\mathring{B}^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})$ is the completion of $\mathcal{S}(\mathbb{R}^n)$ in $B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})$. In this case we only have the dual pairing (5.56) for the Meyer wavelets, because they are elements of $\mathcal{S}(\mathbb{R}^n)$ and for the Daubechies wavelets, because they have a compact support.

In the case $\max(p,q)<\infty$ we can extend this to more general wavelet bases. The wavelets belong to $C^u(\mathbb{R}^n,\kappa)$ and we show under which conditions on $u\in\mathbb{N}_0$ and $\kappa\ge 0$ we have

$$C^u(\mathbb{R}^n,\kappa) \hookrightarrow B^{\sigma_p-s,mloc}_{p'q'}(\mathbb{R}^n,\boldsymbol{w^{-1}}) \ .$$

First of all, we show $C^0(\mathbb{R}^n,\kappa) \hookrightarrow B^{-\alpha_1,mloc}_{p'\infty}(\mathbb{R}^n,\boldsymbol{w^{-1}})$. We use our fixed resolution of unity (3.1), the weight properties and Young's inequality, to see that

$$\begin{aligned}
\left\| f|\, B^{-\alpha_1,mloc}_{p'\infty}(\mathbb{R}^n,\boldsymbol{w^{-1}})\right\| &= \sup_{j\in\mathbb{N}_0} 2^{-\alpha_1 j}\left\| w_j^{-1}(\varphi_j\hat{f})^\vee \right| L_{p'}(\mathbb{R}^n)\Big\| \\
&\le c \sup_{j\in\mathbb{N}_0} \left\| w_0^{-1}(\varphi_j^\vee * f)\right| L_{p'}(\mathbb{R}^n)\Big\| \\
&\le c' \sup_{j\in\mathbb{N}_0} \left\| (w_0^{-1}\varphi_j^\vee) * ((1+|\cdot|)^\alpha f)\right| L_{p'}(\mathbb{R}^n)\Big\| \\
&\le c'' \left\| (1+|\cdot|)^\alpha f|\, L_{p'}(\mathbb{R}^n)\right\| \sup_{j\in\mathbb{N}_0} \left\| \varphi_j^\vee \right| L_1(\mathbb{R}^n)\Big\| \\
&\le c''' \left\| f|\, C^0(\mathbb{R}^n,\kappa)\right\| \ ,
\end{aligned}$$

where the last inequality comes from $\varphi_j=\varphi(2^{-j}\cdot)\in\mathcal{S}(\mathbb{R}^n)$ and that $\kappa>\frac{n}{p'}+\alpha$. Let us assume now that $f\in C^u(\mathbb{R}^n,\kappa)$, then by applying Theorem 2.20 we conclude

$$C^u(\mathbb{R}^n,\kappa) \hookrightarrow B^{u-\alpha_1,mloc}_{p'\infty}(\mathbb{R}^n,\boldsymbol{w^{-1}}) \ .$$

Finally, Theorem 2.30 gives us for $u>\sigma_p-s+\alpha_1$

$$C^u(\mathbb{R}^n,\kappa) \hookrightarrow B^{\sigma_p-s,mloc}_{p'q'}(\mathbb{R}^n,\boldsymbol{w^{-1}}) \ . \tag{5.62}$$

Because of (5.60) and (5.62) the equation (5.59) makes sense at least for $u>\sigma_p-s+\alpha_1$ and $\kappa>\frac{n}{p'}+\alpha$.

Further, we mention that all functions $\{k_{jl}\}$ from Definition 5.22 with $A\ge u$, $B\in\mathbb{N}_0$ arbitrary and $C\ge\kappa$ belong to the space $C^u(\mathbb{R}^n,\kappa)$. So we see that (5.56) is well defined for $A>\sigma_p-s+\alpha_1$ and $C>\alpha+n$, but these conditions will always be fulfilled in the following theorems.

5.3.3 Wavelet isomorphism

We want to use the molecular decomposition obtained in the last section. We assume that $\{\mu_{\nu m}\}$ are $[K, L, M]$ molecules and that the $\{k_{jl}\}$ are the above given functions. Before coming to the theorem we have to prove some fundamental lemmas. First, we have to give estimates of the quantity $|\langle \mu_{\nu m}, k_{jl}\rangle|$. Here $\langle f, g\rangle = \int fg$ denotes the dual pairing which is linear in both entries. The proofs go back to [FrJa90, Appendix B].

Lemma 5.24: *Let $\nu \geq j$, $M > A + n$ and $L \geq A$, then*

$$|\langle \mu_{\nu m}, k_{jl}\rangle| \leq c2^{-(\nu-j)(A+n)}(1 + 2^j|2^{-\nu}m - 2^{-j}l|)^{-\min(M-A-n,C)} . \tag{5.63}$$

Proof: With some substitutions and straightforward calculations we derive

$$|\langle \mu_{\nu m}, k_{jl}\rangle| = 2^{-(\nu-j)n}|(k * g)(2^\nu(2^{-\nu}m + 2^{-j}l))| ,$$

where $g(z) = \overline{\mu_{\nu m}(2^{-\nu}m - 2^{-\nu}z)}$ and $k(x) = 2^{-jn}k_{jl}(2^{-\nu}x + 2^{-j}l)$. One can use the proof of Lemma 5.12. If we look in the proof of this lemma, then we see that we do not need $g_0 \in \mathcal{S}(\mathbb{R}^n)$. We only need the moment conditions on $\mu_{\nu m}$ and a sufficiently strong decay of the derivatives of $k_{j,l}$. □

By interchanging the roles of k_{jl} and $\mu_{\nu m}$ we get using Lemma 5.12 again:

Lemma 5.25: *Let $\nu \leq j$, $C > K + n$ and $B \geq K$, then*

$$|\langle \mu_{\nu m}, k_{jl}\rangle| \leq c2^{-(j-\nu)K}(1 + 2^\nu|2^{-\nu}m - 2^{-j}l|)^{-\min(M,C-K-n)} . \tag{5.64}$$

The next lemma is more or less a discrete version of Lemma 5.15 and is presented in [Kyr03, Lemma 7.4]. Nevertheless, we give a proof because our notation is a bit different.

Lemma 5.26: *Let $\nu \geq j$, $1 \leq p \leq \infty$ and $b_m \in \mathbb{C}$, then*

$$\left(\sum_{l\in\mathbb{Z}^n}\left(\sum_{m\in\mathbb{Z}^n}|b_m|(1 + 2^j|2^{-j}l - 2^{-\nu}m|)^{-R}\right)^p\right)^{1/p} \leq c2^{(\nu-j)\frac{n}{p'}}\left(\sum_{m\in\mathbb{Z}^n}|b_m|^p\right)^{1/p} ,$$

where $R > n$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof: For $k \in \mathbb{Z}^n$ we introduce the quantity

$$\ell^j_\nu(k) = \{m \in \mathbb{Z}^n : Q_{jk} \cap Q_{\nu m} \neq \emptyset\} = \{m \in \mathbb{Z}^n : Q_{\nu m} \subseteq Q_{jk}\} . \tag{5.65}$$

Then we get for $m \in \ell^j_\nu(k)$

$$(1 + |l - k|) \leq c(1 + 2^j|2^{-j}l - 2^{-\nu}m|) . \tag{5.66}$$

So we derive by (5.66) and Minkowski's inequality

$$
\begin{aligned}
&\left(\sum_{l\in\mathbb{Z}^n}\left(\sum_{m\in\mathbb{Z}^n}|b_m|(1+2^j|2^{-j}l-2^{-\nu}m|)^{-R}\right)^p\right)^{1/p}\\
&\qquad=\left(\sum_{l\in\mathbb{Z}^n}\left(\sum_{k\in\mathbb{Z}^n}\sum_{m\in\ell^j_\nu(k)}|b_m|(1+2^j|2^{-j}l-2^{-\nu}m|)^{-R}\right)^p\right)^{1/p}\\
&\qquad\le c\left(\sum_{l\in\mathbb{Z}^n}\left(\sum_{k\in\mathbb{Z}^n}(1+|k-l|)^{-R}\sum_{m\in\ell^j_\nu(k)}|b_m|\right)^p\right)^{1/p}\\
&\qquad= c\left(\sum_{l\in\mathbb{Z}^n}\left(\sum_{u\in\mathbb{Z}^n}(1+|u|)^{-R}\sum_{m\in\ell^j_\nu(u+l)}|b_m|\right)^p\right)^{1/p}\\
&\qquad\le c\sum_{u\in\mathbb{Z}^n}(1+|u|)^{-R}\left(\sum_{l\in\mathbb{Z}^n}\left(\sum_{m\in\ell^j_\nu(u+l)}|b_m|\right)^p\right)^{1/p}.
\end{aligned}
$$

Finally Hölder's inequality and $\operatorname{card}\ell^j_\nu(u+l)\sim 2^{(\nu-j)n}$ give us

$$
\begin{aligned}
&\left(\sum_{l\in\mathbb{Z}^n}\left(\sum_{m\in\mathbb{Z}^n}|b_m|(1+2^j|2^{-j}l-2^{-\nu}m|)^{-R}\right)^p\right)^{1/p}\\
&\qquad\le c2^{(\nu-j)\frac{n}{p'}}\sum_{u\in\mathbb{Z}^n}(1+|u|)^{-R}\left(\sum_{l\in\mathbb{Z}^n}\sum_{m\in\ell^j_\nu(u+l)}|b_m|^p\right)^{1/p}\\
&\qquad\le c'2^{(\nu-j)\frac{n}{p'}}\left(\sum_{m\in\mathbb{Z}^n}|b_m|^p\right)^{1/p},
\end{aligned}
$$

where the last inequality is due to $\operatorname{card}\{l\in\mathbb{Z}^n : Q_{\nu m}\subseteq Q_{jl}+2^{-j}u\}\sim 1$ and $R>n$. □

The last lemma is of the same type, only with $\nu\le j$. The proof is in principle very similar to the one of Lemma 5.26.

Lemma 5.27: *Let $\nu\le j$, $1\le p\le\infty$ and $b_m\in\mathbb{C}$, then*

$$
\left(\sum_{l\in\mathbb{Z}^n}\left(\sum_{m\in\mathbb{Z}^n}|b_m|(1+2^\nu|2^{-j}l-2^{-\nu}m|)^{-R}\right)^p\right)^{1/p}\le c2^{(j-\nu)\frac{n}{p}}\left(\sum_{m\in\mathbb{Z}^n}|b_m|^p\right)^{1/p},
$$

where $R>n$.

Now, we are ready to state the first theorem, which gives us one direction of the wavelet decomposition. We define $k(f)=\{k_{jl}(f) : j\in\mathbb{N}_0, l\in\mathbb{Z}^n\}$.

Theorem 5.28: *Let $w \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $s \in \mathbb{R}$ and $0 < p,q \leq \infty$. Further, let $\{k_{jl}\}$ be as in Definition 5.22 with $C > 0$ large enough and $A, B \in \mathbb{N}_0$ with*

$$A > \sigma_p - s + \alpha_1 \ , \qquad B > s + \alpha_2. \tag{5.67}$$

Then for some $c > 0$ and all $f \in B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$,

$$\left\| k(f) | \, b^{s,mloc}_{pq}(\boldsymbol{w}) \right\| \leq c \left\| f | \, B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right\| \ .$$

Proof: We have $f \in B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ and we can use the atomic decomposition theorem (Corollary 5.18) and get

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m} \ ,$$

where $\{a_{\nu m}\}$ are $[K, L]$ atoms with $K = B > s + \alpha_2$ and $L = A > \sigma_p - s + \alpha_1$. We want to show

$$\left\| k(f) | \, b^{s,mloc}_{pq}(\boldsymbol{w}) \right\| \leq c \left\| \lambda | \, b^{s,mloc}_{pq}(\boldsymbol{w}) \right\| \leq c' \left\| f | \, B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right\| \ , \tag{5.68}$$

where the last inequality comes from the atomic decomposition theorem. We recall that atoms have compact support and that they are $[K, L, M]$ molecules for every $M > 0$. We split

$$f = f_j + f^j = \sum_{\nu=0}^{j} \cdots + \sum_{\nu=j+1}^{\infty} \cdots \qquad \text{and get}$$

$$k_{jl}(f) = \int_{\mathbb{R}^n} k_{jl}(y) f(y) dy = \int_{\mathbb{R}^n} k_{jl}(y) f_j(y) dy + \int_{\mathbb{R}^n} k_{jl}(y) f^j(y) dy \ .$$

Let $\nu \leq j$ and $1 \leq p \leq \infty$, then we estimate by Minkowski's inequality and Lemma 5.25

$$\begin{aligned}
&\left(\sum_{l \in \mathbb{Z}^n} \left| \int_{\mathbb{R}^n} k_{jl}(y) f_j(y) dy \right|^p w_j^p(2^{-j} l) \right)^{1/p} \\
&\quad \leq \sum_{\nu=0}^{j} \left(\sum_{l \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}| w_j(2^{-j} l) | \langle k_{jl}, a_{\nu m} \rangle | \right)^p \right)^{1/p} \\
&\quad \leq c \sum_{\nu=0}^{j} 2^{-(j-\nu)(B-\alpha_2)} \left(\sum_{l \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}| w_\nu(2^{-\nu} m) (1 + 2^\nu |2^{-\nu} m - 2^{-j} l|)^{-C+\alpha} \right)^p \right)^{1/p} && (5.69) \\
&\quad \leq c' \sum_{\nu=0}^{j} 2^{-(j-\nu)(K-\alpha_2-\frac{n}{p})} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p w_\nu^p(2^{-\nu} m) \right)^{1/p} \ , && (5.70)
\end{aligned}$$

where in the last step we used Lemma 5.27 with $R = C - \alpha > n$.
In the case $0 < p < 1$ we have also the estimate (5.69) and get by $\ell_p \hookrightarrow \ell_1$

$$\begin{aligned}\left(\sum_{l\in\mathbb{Z}^n}\left|\int\limits_{\mathbb{R}^n} k_{jl}(y)f_j(y)dy\right|^p w_j^p(2^{-j}l)\right)^{1/p}\\ \leq c\sum_{\nu=0}^{j} 2^{-(j-\nu)(B-\alpha_2)}\left(\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|^p w_\nu^p(2^{-\nu}m)\sum_{l\in\mathbb{Z}^n}(1+2^\nu|2^{-\nu}m-2^{-j}l|)^{-p(C-\alpha)}\right)^{1/p}.\end{aligned} \tag{5.71}$$

A direct calculation shows that $\sum_{l\in\mathbb{Z}^n}(1+2^\nu|2^{-\nu}m-2^{-j}l|)^{-p(C-\alpha)} \leq c2^{(j-\nu)n}$ for $C > \frac{n}{p}+\alpha$. Therefore, we obtain from (5.70) and (5.71)

$$\begin{aligned}2^{j(s-\frac{n}{p})}\left(\sum_{l\in\mathbb{Z}^n}\left|\int\limits_{\mathbb{R}^n} k_{jl}(y)f_j(y)dy\right|^p w_j^p(2^{-j}l)\right)^{1/p}\\ \leq c\sum_{\nu=0}^{j} 2^{\nu(s-\frac{n}{p})}2^{-(j-\nu)\zeta}\left(\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|^p w_\nu^p(2^{-\nu}m)\right)^{1/p},\end{aligned} \tag{5.72}$$

where $\zeta = B - s - \alpha_2 > 0$.
Let us consider the case $\nu > j$ and $1 \leq p \leq \infty$. Then we derive similar to the first case with Minkowski's inequality and Lemma 5.24

$$\begin{aligned}&\left(\sum_{l\in\mathbb{Z}^n}\left|\int\limits_{\mathbb{R}^n} k_{jl}(y)f^j(y)dy\right|^p w_j^p(2^{-j}l)\right)^{1/p}\\ &\leq \sum_{\nu=j+1}^{\infty}\left(\sum_{l\in\mathbb{Z}^n}\left(\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|w_j(2^{-j}l)|\langle k_{jl},a_{\nu m}\rangle|\right)^p\right)^{1/p}\\ &\leq c\sum_{\nu=j+1}^{\infty}2^{-(\nu-j)(A+n-\alpha_1)}\left(\sum_{l\in\mathbb{Z}^n}\left(\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|w_\nu(2^{-\nu}m)(1+2^j|2^{-\nu}m-2^{-j}l|)^{-C+A+n+\alpha}\right)^p\right)^{1/p}\\ &\leq c'\sum_{\nu=j+1}^{\infty}2^{-(\nu-j)(A-\alpha_1+\frac{n}{p})}\left(\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|^p w_\nu^p(2^{-\nu}m)\right)^{1/p},\end{aligned} \tag{5.73}$$

where we used Lemma 5.26 with $R = C - A - n - \alpha > n$ in the last step. We obtain now with $\rho = A + s - \alpha_1 > 0$

$$2^{j(s-\frac{n}{p})}\left(\sum_{l\in\mathbb{Z}^n}\left|\int_{\mathbb{R}^n} k_{jl}(y)f^j(y)dy\right|^p w_j^p(2^{-j}l)\right)^{1/p}$$

$$\leq c\sum_{\nu=j+1}^{\infty} 2^{\nu(s-\frac{n}{p})}2^{-(\nu-j)\rho}\left(\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|^p w_\nu^p(2^{-\nu}m)\right)^{1/p} . \qquad (5.74)$$

For $0 < p < 1$ we have again (5.73), use $\ell_p \hookrightarrow \ell_1$ and obtain

$$\left(\sum_{l\in\mathbb{Z}^n}\left|\int_{\mathbb{R}^n} k_{jl}(y)f^j(y)dy\right|^p w_j^p(2^{-j}l)\right)^{1/p}$$

$$\leq c\sum_{\nu=j+1}^{\infty} 2^{-(\nu-j)(A+n-\alpha_1)}\left(\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|^p w_\nu^p(2^{-\nu}m)\sum_{l\in\mathbb{Z}^n}(1+2^j|2^{-\nu}m-2^{-j}l|)^{-p(C-A-n-\alpha)}\right)^{1/p} .$$

The sum over $l \in \mathbb{Z}^n$ is bounded by a constant for $C > \frac{n}{p} + n + A + \alpha$. Hence, we get

$$2^{j(s-\frac{n}{p})}\left(\sum_{l\in\mathbb{Z}^n}\left|\int_{\mathbb{R}^n} k_{jl}(y)f^j(y)dy\right|^p w_j^p(2^{-j}l)\right)^{1/p}$$

$$\leq c\sum_{\nu=j+1}^{\infty} 2^{\nu(s-\frac{n}{p})}2^{-(\nu-j)\tilde{\rho}}\left(\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|^p w_\nu^p(2^{-\nu}m)\right)^{1/p} , \qquad (5.75)$$

where $\tilde{\rho} = A + s - \alpha_1 - (n/p - n) > 0$.
Now the result (5.68) can be obtained from (5.72), (5.74) and (5.75) by standard arguments. □

Remark 5.29: As shown in the proof it is enough to assume

$$C > \max(A, B) + n\left(1 + \frac{1}{\min(1,p)}\right) + \alpha .$$

That B appears above comes from the conditions in Lemma 5.25.

We come to the wavelet decomposition theorem. Let us assume that

$$\psi_M \in C^k(\mathbb{R}^n) \quad \text{and} \quad \psi_F \in C^k(\mathbb{R}^n) \qquad (5.76)$$

are the real compactly supported Daubechies wavelets from Theorem 5.19, with

$$\int_{\mathbb{R}^n} x^\beta \psi_M(x)dx = 0 \qquad \text{for } |\beta| < k. \qquad (5.77)$$

By the tensor product procedure (5.53), we have that $\{\Psi^{\nu}_{G,m} : \nu \in \mathbb{N}_0, G \in G^{\nu}$ and $m \in \mathbb{Z}^n\}$ is an orthonormal basis in $L_2(\mathbb{R}^n)$.
Before coming to the theorem we clarify the convergence of

$$\sum_{\nu=0}^{\infty} \sum_{G \in G^{\nu}} \sum_{m \in \mathbb{Z}^n} \lambda^{\nu}_{Gm} 2^{-\nu\frac{n}{2}} \Psi^{\nu}_{Gm} \qquad \text{with } \lambda \in \tilde{b}^{s,mloc}_{pq}(\boldsymbol{w}) \ . \tag{5.78}$$

We say that a series converges unconditionally, if any rearrangement of the series also converges to the same outcome. We know that $\{2^{-\nu\frac{n}{2}}\Psi^{\nu}_{Gm}\}$ are $[k,k,M]$ molecules for every $M > 0$ and therefore we have the unconditional convergence of (5.78) in $\mathcal{S}'(\mathbb{R}^n)$ from Lemma 5.7 with $k > \sigma_p - s + \alpha_1$.
Moreover, the following proof shows the unconditional convergence of (5.78) for $0 < p < \infty$ and $0 < q < \infty$ in $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$. If $0 < p < \infty$ and $0 < q \le \infty$ then we have unconditional convergence in $B^{\sigma,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ with $\sigma < s$. For general $0 < p, q \le \infty$ we get the unconditional convergence in $B^{\sigma,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{\varrho})$ where $\sigma < s$ and ϱ is an admissible weight sequence with $\frac{\varrho_\nu(x)}{w_\nu(x)} \to 0$ for $|x| \to \infty$.
Theorem 5.30: *Let $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $s \in \mathbb{R}$, $0 < p, q \le \infty$ and*

$$k > \max(\sigma_p - s + \alpha_1, s + \alpha_2) \tag{5.79}$$

in (5.76) and (5.77). Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then $f \in B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ if, and only if, it can be represented as

$$f = \sum_{\nu=0}^{\infty} \sum_{G \in G^{\nu}} \sum_{m \in \mathbb{Z}^n} \lambda^{\nu}_{Gm} 2^{-\nu\frac{n}{2}} \Psi^{\nu}_{Gm} \qquad \textit{with } \lambda \in \tilde{b}^{s,mloc}_{pq}(\boldsymbol{w}) \ , \tag{5.80}$$

with unconditional convergence in $\mathcal{S}'(\mathbb{R}^n)$ and in any $B^{\sigma,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{\varrho})$ with $\sigma < s$ and $\frac{\varrho_\nu(x)}{w_\nu(x)} \to 0$ for $|x| \to \infty$. The representation (5.80) is unique,

$$\lambda^{\nu}_{Gm} = \lambda^{\nu}_{Gm}(f) = 2^{\nu\frac{n}{2}} \langle f, \Psi^{\nu}_{Gm} \rangle \tag{5.81}$$

and

$$I : f \mapsto \{2^{\nu\frac{n}{2}} \langle f, \Psi^{\nu}_{Gm} \rangle\} \tag{5.82}$$

is an isomorphic map from $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ onto $\tilde{b}^{s,mloc}_{pq}(\boldsymbol{w})$. Moreover, if in addition $\max(p,q) < \infty$ then $\{\Psi^{\nu}_{Gm}\}$ is in unconditional basis in $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$.

Proof: First Step: Let $f \in \mathcal{S}'(\mathbb{R}^n)$ be given by (5.80). Then by the support properties we have that $\{2^{-\nu\frac{n}{2}}\Psi^{\nu}_{Gm}\}$ are $[k,k,M]$ molecules for every $M > 0$. From Theorem 5.17 and (5.79) we obtain $f \in B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ and

$$\left\| f | \, B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \right\| \le c \left\| \lambda | \, \tilde{b}^{s,mloc}_{pq}(\boldsymbol{w}) \right\| \tag{5.83}$$

with $c > 0$ independent of $\lambda \in \tilde{b}^{s,mloc}_{pq}(\boldsymbol{w})$.
Second Step: Let $f \in B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ then we can apply Theorem 5.28 with $k_{\nu m} =$

$2^{\nu\frac{n}{2}}\Psi^{\nu}_{Gm}$. Since all conditions on $k_{\nu m}$ are fulfilled by (5.79) and the compact support of the wavelets we get

$$\left\|\lambda(f)|\,\tilde{b}^{s,mloc}_{pq}(\boldsymbol{w})\right\| \le c\left\|f|\,B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})\right\| \ . \tag{5.84}$$

Third Step: For $\max(p,q)<\infty$ we get the unconditional convergence of (5.80) in $B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})$ by (5.83) and the properties of the sequence spaces $\tilde{b}^{s,mloc}_{pq}(\boldsymbol{w})$.
Let $p<\infty$ and $q=\infty$, then we get the convergence in $B^{\sigma,mloc}_{p\infty}(\mathbb{R}^n,\boldsymbol{w})$ for all $\sigma<s$ in using (5.83) again and Hölder's inequality.
To obtain the convergence for $p=\infty$ we have to compensate the behavior at infinity by introducing a weaker weight sequence $\boldsymbol{\varrho}$ with $\frac{\varrho_\nu(x)}{w_\nu(x)}\to 0$ for $|x|\to\infty$. Then we get the unconditional convergence in $B^{\sigma,mloc}_{\infty q}(\mathbb{R}^n,\boldsymbol{\varrho})$ with $\sigma<s$ as in the previous case.
A simple example of such a weaker weight sequence is given for every $\varepsilon>0$ by

$$\varrho_\nu(x)=(1+2^\nu|x|)^{-\varepsilon}w_\nu(x) \tag{5.85}$$

which belongs to $\mathcal{W}^{\alpha+\varepsilon}_{\alpha_1+\varepsilon,\alpha_2}$.
Fourth Step: We want to prove now the uniqueness of the coefficients. We define

$$g=\sum_{\nu=0}^{\infty}\sum_{G\in G^\nu}\sum_{m\in\mathbb{Z}^n}\lambda^{\nu}_{Gm}2^{-\nu\frac{n}{2}}\Psi^{\nu}_{Gm} \tag{5.86}$$

where λ^ν_{Gm} is given by (5.81). We want to show that $g=f$, or

$$\langle\, g,\varphi\rangle=\langle\, f,\varphi\rangle \quad \text{for every } \varphi\in\mathcal{S}(\mathbb{R}^n). \tag{5.87}$$

From the first step we have $g\in B^{s,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{w})$. The third step tells us that (5.86) converges at least in $B^{\sigma,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{\varrho})$ for all $\sigma<s$ and $\boldsymbol{\varrho}$ is given by (5.85) for some $\varepsilon>0$. Since $k>\sigma_p-s+\alpha_1$ we can find $\sigma<s$ and $\varepsilon>0$ such that $\Psi^{\nu'}_{G'm'}$ still belongs to the dual space $(B^{\sigma,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{\varrho}))'$ (that means $k>\sigma_p-\sigma+\alpha_1+\varepsilon$). Because of the convergence in $B^{\sigma,mloc}_{pq}(\mathbb{R}^n,\boldsymbol{\varrho})$,the dual pairing and the orthonormality of $\{\Psi^\nu_{Gm}\}$ in $L_2(\mathbb{R}^n)$ we get

$$\left\langle\, g,\Psi^{\nu'}_{G'm'}\right\rangle=\left\langle\, f,\Psi^{\nu'}_{G'm'}\right\rangle \ . \tag{5.88}$$

This holds also for finite linear combinations of $\Psi^{\nu'}_{G'm'}$. For a function $\varphi\in\mathcal{S}(\mathbb{R}^n)$ we have the unique $L_2(\mathbb{R}^n)$ representation

$$\varphi=\sum_{\nu,G,m}2^{-\nu\frac{n}{2}}\left\langle\,\varphi,\Psi^\nu_{Gm}\right\rangle\Psi^\nu_{Gm} \ . \tag{5.89}$$

Since $\mathcal{S}(\mathbb{R}^n)$ is a subspace in every Besov space considered this representation converges in $(B^{\sigma,mloc}_{p,q}(\mathbb{R}^n,\boldsymbol{w}))'$ and we get by (5.88)

$$\langle\, g,\varphi\rangle=\langle\, f,\varphi\rangle \ . \tag{5.90}$$

Final Step: Hence, $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ if, and only if, it can be represented by (5.80). This representation is unique with coefficients (5.81). By (5.83), (5.86), with $g = f$, and (5.84) it follows

$$\left\| \lambda(f) | \tilde{b}^{s,mloc}_{pq}(\boldsymbol{w}) \right\| \sim \| f | B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \| \ . \tag{5.91}$$

Hence I in (5.82) is an isomorphic map from $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ into $\tilde{b}^{s,mloc}_{pq}(\boldsymbol{w})$. It remains to prove that this map is onto. Let $\lambda \in \tilde{b}^{s,mloc}_{pq}(\boldsymbol{w})$. Then it follows by the above considerations that

$$f = \sum_{\nu,G,m} \lambda^{\nu}_{Gm} 2^{-\nu\frac{n}{2}} \Psi^{\nu}_{Gm} \quad \in B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w}) \ .$$

By the same reasoning as in the fourth step this representation is unique and $\lambda^{\nu}_{Gm} = \lambda^{\nu}_{Gm}(f)$. This proves that I is a map onto. $\square$

To proof the wavelet decomposition with Daubechies wavelets we only used the atomic decomposition theorem.
Now, we present a wavelet decomposition theorem with the help of the Meyer wavelets, described in Theorem 5.19. We have $\psi_M,\ \psi_F \in \mathcal{S}(\mathbb{R}^n)$ and we have infinitely many moment conditions on ψ_M. But we lose the compact support property for the wavelets. Here we need to use our molecular decomposition (Theorem 5.17). The proof is the same as in Theorem 5.30. We use again our wavelets once as molecules and once as kernels from Definition 5.22 where the technicalities get easier because A, B, C are infinite.

Theorem 5.31: *Let $\{\Psi^{\nu}_{Gm}\}$ be the Meyer wavelets according to Theorem 5.19. Further, let $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $s \in \mathbb{R}$, $0 < p, q \le \infty$ and let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then $f \in B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ if, and only if, it can be represented as*

$$f = \sum_{\nu=0}^{\infty} \sum_{G \in G^{\nu}} \sum_{m \in \mathbb{Z}^n} \lambda^{\nu}_{Gm} 2^{-\nu\frac{n}{2}} \Psi^{\nu}_{Gm} \qquad \text{with } \lambda \in \tilde{b}^{s,mloc}_{pq}(\boldsymbol{w}) \ , \tag{5.92}$$

with unconditional convergence in $\mathcal{S}'(\mathbb{R}^n)$ and in any $B^{\sigma,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{\varrho})$ with $\sigma < s$ and $\frac{\varrho_\nu(x)}{w_\nu(x)} \to 0$ for $|x| \to \infty$. The representation (5.80) is unique,

$$\lambda^{\nu}_{Gm} = \lambda^{\nu}_{Gm}(f) = 2^{\nu\frac{n}{2}} \left\langle f, \Psi^{\nu}_{Gm} \right\rangle \tag{5.93}$$

and

$$I : f \mapsto \{2^{\nu\frac{n}{2}} \left\langle f, \Psi^{\nu}_{Gm} \right\rangle\} \tag{5.94}$$

is an isomorphic map from $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ onto $\tilde{b}^{s,mloc}_{pq}(\boldsymbol{w})$. Moreover, if in addition $\max(p, q) < \infty$ then $\{\Psi^{\nu}_{Gm}\}$ is in unconditional basis in $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$.

Remark 5.32: The wavelet characterization of $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ is not restricted to the two wavelet systems presented in Theorem 5.19. The proof of Theorem 5.30 also applies to all wavelet systems $\{\Psi^\nu_{Gm}\}$ which satisfy that $2^{-\nu\frac{n}{2}}\{\Psi^\nu_{Gm}\}$ are $[K,K,M]$ molecules with

$$K > \max(\sigma_p - s + \alpha_1, s + \alpha_2) \quad \text{and} \quad M > K + n\max(2, 1+1/p) + 2\alpha \ , \tag{5.95}$$

where the condition on M is, presumably, not sharp.
The proofs can easily be extended to biorthogonal wavelet bases (see [Kyr03] for details).

5.3.4 Wavelet decomposition of $\mathbf{B^{s,s'}_{pq}}(\mathbb{R}^{\mathbf{n}}, \mathbf{U})$

We recall the spaces $B^{s,s'}_{pq}(\mathbb{R}^n, U)$ (Definition 2.21) which have for a bounded subset $U \subset \mathbb{R}^n$ and $s' \in \mathbb{R}$ the weight sequence of 2-microlocal weights

$$w_\nu(x) = (1 + 2^\nu \operatorname{dist}(x, U))^{s'} \ . \tag{5.96}$$

From Example 2.5 we know that $\boldsymbol{w} \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$ for $|s'| \le \alpha$ and $-\min(0,s') \le \alpha_1$, $\max(0,s') \le \alpha_2$. Our spaces $B^{s,s'}_{pq}(\mathbb{R}^n, U)$ are defined as usual as the spaces $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ with the special 2-microlocal weight function (5.96) for $\boldsymbol{w}$. The corresponding sequence spaces are defined by the norm

$$\left\| \lambda |\, \tilde{b}^{s,s'}_{pq}(U) \right\| = \left(\sum_{\nu=0}^\infty 2^{\nu(s-n/p)q} \sum_{G\in G^\nu} \left(\sum_{m\in\mathbb{Z}^n} |\lambda^\nu_{Gm}|^p (1 + 2^\nu \operatorname{dist}(2^{-\nu}m, U))^{ps'} \right)^{q/p} \right)^{1/q} .$$

Now, we can adopt everything from Theorem 5.30 and our wavelet decomposition for the 2-microlocal Besov spaces with respect to the Daubechies wavelets follows.

Corollary 5.33: *Let $U \subset \mathbb{R}^n$ bounded, $s' \in \mathbb{R}$ and $\boldsymbol{w}$ as in* (5.96)*. Further, let $s \in \mathbb{R}$, $0 < p, q \le \infty$ and*

$$k > \max(\sigma_p - s - \min(0,s'), s + \max(0,s')) \tag{5.97}$$

in (5.76) *and* (5.77)*. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then $f \in B^{s,s'}_{pq}(\mathbb{R}^n, U)$ if, and only if, it can be represented as*

$$f = \sum_{\nu=0}^\infty \sum_{G\in G^\nu} \sum_{m\in\mathbb{Z}^n} \lambda^\nu_{Gm} 2^{-\nu\frac{n}{2}} \Psi^\nu_{Gm} \qquad \textit{with } \lambda \in \tilde{b}^{s,s'}_{pq}(U) \ , \tag{5.98}$$

with unconditional convergence in $\mathcal{S}'(\mathbb{R}^n)$ and in any $B^{t,t'}_{pq}(\mathbb{R}^n, U)$ with $t < s$ and $t' < s'$. The representation (5.98) *is unique,*

$$\lambda^\nu_{Gm} = \lambda^\nu_{Gm}(f) = 2^{\nu\frac{n}{2}} \langle f, \Psi^\nu_{Gm} \rangle \tag{5.99}$$

and

$$I : f \mapsto \{2^{\nu\frac{n}{2}} \langle f, \Psi^\nu_{Gm} \rangle\} \tag{5.100}$$

is an isomorphic map from $B^{s,s'}_{pq}(\mathbb{R}^n, U)$ onto $\tilde{b}^{s,s'}_{pq}(U)$. Moreover, if in addition $\max(p,q) < \infty$ then $\{\Psi^\nu_{Gm}\}$ is in unconditional basis in $B^{s,s'}_{pq}(\mathbb{R}^n, U)$.

Let us say a few words about the convergence of (5.98). As in Theorem 5.30 we have unconditional convergence in $B^{s,s'}_{pq}(\mathbb{R}^n, U)$ for $\max(p, q) < \infty$ and arbitrary $U \in \mathbb{R}^n$, not necessarily bounded. For $0 < p < \infty$ and $0 < q \leq \infty$ we have unconditional convergence in $B^{t,s'}_{pq}(\mathbb{R}^n, U)$ with $s > t$ and also arbitrary U. Only in the case of $p = \infty$ we need as an additional assumption that $U \subset \mathbb{R}^n$ has to be bounded to get the unconditional convergence in $B^{t,t'}_{pq}(\mathbb{R}^n, U)$ for all $s > t$ and $s' > t'$.

Remark 5.34: For $p \geq 1$ we have $\sigma_p = 0$ in condition (5.97). Now we can rewrite this condition as

$$k > \max(|s|, |s + s'|) \ . \tag{5.101}$$

This is almost the same condition

$$k > \max(\max(s, s + s'), \max(-(s + n), -(s + s')))$$

used in [Mey97, p.64, p.67] in the case $U = \{x_0\}$ and $p = q = \infty$ in connection with the orthonormal Daubechies wavelets.

There is no doubt that we can rewrite Theorem 5.31, where we get the wavelet decomposition with Meyer wavelets.

Corollary 5.35: *Let $U \subset \mathbb{R}^n$ bounded, $s' \in \mathbb{R}$, $\boldsymbol{w}$ as in* (5.96) *and let $\{\Psi^\nu_{Gm}\}$ be the Meyer wavelets according to Theorem 5.19. Further, let $s \in \mathbb{R}$, $0 < p, q \leq \infty$ and let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then $f \in B^{s,s'}_{pq}(\mathbb{R}^n, U)$ if, and only if, it can be represented as*

$$f = \sum_{\nu=0}^{\infty} \sum_{G \in G^\nu} \sum_{m \in \mathbb{Z}^n} \lambda^\nu_{Gm} 2^{-\nu\frac{n}{2}} \Psi^\nu_{Gm} \qquad \textit{with } \lambda \in \tilde{b}^{s,s'}_{pq}(U) \ , \tag{5.102}$$

with unconditional convergence in $\mathcal{S}'(\mathbb{R}^n)$ and in any $B^{t,t'}_{pq}(\mathbb{R}^n, U)$ with $t < s$ and $t' < s'$. The representation (5.102) *is unique,*

$$\lambda^\nu_{Gm} = \lambda^\nu_{Gm}(f) = 2^{\nu\frac{n}{2}} \langle f, \Psi^\nu_{Gm} \rangle \tag{5.103}$$

and

$$I : f \mapsto \{2^{\nu\frac{n}{2}} \langle f, \Psi^\nu_{Gm} \rangle\} \tag{5.104}$$

is an isomorphic map from $B^{s,s'}_{pq}(\mathbb{R}^n, U)$ onto $\tilde{b}^{s,s'}_{pq}(U)$. Moreover, if in addition $\max(p, q) <$ ∞ *then $\{\Psi^\nu_{Gm}\}$ is in unconditional basis in $B^{s,s'}_{pq}(\mathbb{R}^n, U)$.*

Remark 5.36: According to Remark 5.32 we can also characterize $B^{s,s'}_{pq}(\mathbb{R}^n, U)$ by other wavelet bases $\{\Psi^\nu_{Gm}\}$, where $\{\Psi^\nu_{Gm}\}$ are $[K, K, M]$ molecules with the condition (5.95) on K and M.

Remark 5.37: These two corollaries are a generalization of the known wavelet decompositions for 2-microlocal spaces $C^{s,s'}_{x_0}(\mathbb{R}^n)$.
The wavelet decomposition for the spaces $C^{s,s'}_{x_0}(\mathbb{R}^n)$ and $H^{s,s'}_{x_0}(\mathbb{R}^n)$ has been presented by Jaffard in [Ja91, Theorem 2]. There exists also a wavelet characterization for $B^{s,s'}_{pq}(\mathbb{R}^n, 0)$ with $p, q \geq 1$ in [Xu96, Theorem 2.13] by Xu. Similar conditions on the smoothness of the Daubechies wavelets as (5.101), sometimes better sometimes worse, appear in many papers [JaMey96], [LVSeu04] and [Xu96] but are not really explained.

6 Application of the wavelet decomposition theorem

In this chapter we apply the wavelet characterization of the spaces $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ to get some results which are already known for $C^{s,s'}_{x_0}(\mathbb{R}^n)$ and to give sharp versions of embeddings.

6.1 Pseudodifferential operators

In this section we show that the 2-microlocal spaces $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ are invariant under the action of classical pseudodifferential operators of order 0. The corresponding results have already been shown in [Ja91] for $C^{s,s'}_{x_0}(\mathbb{R}^n)$ and in [And97] for $F^{s,s'}_{p,q}(\mathbb{R}^n, x_0)$ with $1 \leq p, q \leq \infty$.
As in [Ja91] we introduce operators T belonging to the class $Op(M^\gamma)$ of Calderon-Zygmund operators, which were introduced in [MeyCoi97].
A linear and continuous operator $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ possesses by the Schwartz kernel theorem a distribution kernel K in $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\langle Tf, g \rangle = \langle K, f \otimes g \rangle \qquad \text{for all } f, g \in \mathcal{S}(\mathbb{R}^n).$$

The distribution K is called the kernel of T. An operator T belongs to the class O^γ for $\gamma > 0$ if its distribution kernel K is, off the main diagonal $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}$, continuous and satisfies

$$|D^\beta K(x,y)| \leq c|x-y|^{-n-|\beta|} \quad \text{for } |\beta| < \gamma$$

and, with $\gamma = m + r$, where $m \in \mathbb{N}_0$ and $r \in (0,1]$

$$|D^\beta K(x,y) - D^\beta K(x',y)| \leq c\frac{|x-x'|^r}{|x-y|^{n+\gamma}} \qquad \text{for } |x-x'| \leq \frac{1}{2}|x-y|$$
$$|D^\beta K(x,y) - D^\beta K(x,y')| \leq c\frac{|y-y'|^r}{|x-y|^{n+\gamma}} \qquad \text{for } |y-y'| \leq \frac{1}{2}|x-y|$$

for $|\beta| = m$. Moreover, T has to be bounded on $L_2(\mathbb{R}^n)$ and $T(X^\beta) = T^*(X^\beta) = 0$ for $|\beta| \leq \gamma$ (see [MeyCoi97] for definition of $T(X^\beta)$). Then $Op(M^\gamma) = \bigcup_{\gamma' > \gamma} O^{\gamma'}$.
The class $Op(M^\gamma)$ can be characterized by decay conditions on the terms of the matrix $(\langle T\Psi_\lambda, \Psi_{\lambda'} \rangle)$ with respect to a wavelet basis $\{\Psi_\lambda\}$ where $\Psi_\lambda = \Psi^\nu_{Gm}$. Therefore, let $\{\Psi^\nu_{Gm}\}$ be the Daubechies wavelets with $k > \gamma$ then the following holds (see [MeyCoi97, Chapter 8]).

Proposition 6.1: *An operator T belongs to $Op(M^\gamma)$ if, and only if, its matrix coefficients satisfy*

$$\left|\left\langle T\Psi^\nu_{Gm}, \Psi^j_{Gl} \right\rangle\right| \leq c\frac{2^{-|j-\nu|(\gamma'+n/2)}}{1+(j-\nu)^2}\left(1 + \frac{|2^{-\nu}m - 2^{-j}l|}{2^{-\nu}+2^{-j}}\right)^{-n-\gamma'}, \tag{6.1}$$

for some $\gamma' > \gamma$.

Remark 6.2: It is also possible to take the Meyer wavelets in the above Proposition. The conditions on the wavelet basis are that it has to be in $C^k(\mathbb{R}^n)$ and satisfies (5.77) with $k > \gamma$ and the wavelets have to be sufficiently strong decaying (see [MeyCoi97] and [Ja06, Theorem1]).

The condition in (6.1) can be reformulated as

$$2^{(j-\nu)n/2} \left|\left\langle T\Psi^{\nu}_{Gm}, \Psi^{j}_{Gl}\right\rangle\right| \leq c \begin{cases} 2^{-(\nu-j)(\gamma'+n)}(1+2^j|2^{-j}l-2^{-\nu}m|)^{-n-\gamma'}, & \text{for } \nu \geq j \\ 2^{-(j-\nu)\gamma'}(1+2^{\nu}|2^{-j}l-2^{-\nu}m|)^{-n-\gamma'}, & \text{for } \nu \leq j. \end{cases} \tag{6.2}$$

That means that our matrix is almost diagonal (in the meaning of [FrJa90]) and we can easily adopt the proof of Theorem 5.28. In this proof we take instead of the atomic representation of $f \in B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ the wavelet characterization of f with respect to the Daubechies wavelets with $k > \gamma > \max(s+\alpha_2, \sigma_p - s + \alpha_1)$. Then $2^{-\nu n/2}\Psi^{\nu}_{Gm}$ plays the role of the molecules $\mu_{\nu m}$ and $2^{jn/2}\Psi^{j}_{Gl}$ replace the kernels k_{jl} in this proof.
Now, as in (5.68) we take $T(f)$ as replacement for $k(f)$ and condition (6.2) instead of Lemmas 5.24 and 5.25.
If one compares the conditions on the decay in (6.2) we realize that $\gamma + n$ is first equal to C for $\nu \geq j$ and later for $j \geq \nu$ to $C - A - n$ in the proof of Theorem 5.28. In both cases we obtain the last condition $\gamma > \sigma_p + \alpha$ and we receive the following.

Theorem 6.3: *Let $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. If f belongs to $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ and T belongs to $Op(M^{\gamma})$ with*

$$\gamma > \max(s+\alpha_2, \sigma_p - s + \alpha_1, \sigma_p + \alpha)$$

then Tf belongs to $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$.

If we keep the last theorem in mind, we get the following interpretation. The position of points of regularity of a function is essentially preserved under the action of singular integral operators, such as the Hilbert transform or the Riesz transforms $R_j = -i\frac{\partial}{\partial x_j}(-\Delta)^{-1/2}$.
Moreover, since every classical pseudodifferential operator belonging to $S^0_{1,0}$ is a sum of an operator in $Op(M^{\gamma})$ and a regularizing operator ([MeyCoi97, Chapter 7]), we conclude that the 2-microlocal spaces $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ are invariant under the action of pseudodifferential operators out of $S^0_{1,0}$.
We can state now a regularity result for solutions of elliptic partial differential equations. From Calderon and Zygmund [CaZy57] we know that the inverse of an elliptic operator is a product of a fractional integration and a pseudodifferential operator of order 0. The latter one is a sum of an operator in $Op(M^{\gamma})$ and a regularizing operator and the first one is a lift operator (Theorem 2.19). Using this fact together with the last theorem, we obtain.

Theorem 6.4: *Let $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. Let Λ be an elliptic partial differential operator of order m, with smooth coefficients. If $\Lambda f = g$ and g belongs to $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$, then f belongs to $B^{s+m,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$.*

6.2 Spaces of varying smoothness

In this section we discuss the connection of $B^{s,s'}_{pq}(\mathbb{R}^n, U)$ to the class of spaces of varying smoothness from [Schn07]. Therefore we fix $U \subset \mathbb{R}^n$ as a compact set. First of all, we need to define the class of function spaces of varying smoothness. These spaces are some kind of Besov spaces where the smoothness parameter s is not a fixed real number, but gets replaced by a smoothness function $\mathbb{S} : \mathbb{R}^n \to \mathbb{R}$.

Definition 6.5: *A real-valued function $\mathbb{S} : x \mapsto s(x)$ on $\mathbb{R}^n$ is called lower semi-continuous, if for any $t \in \mathbb{R}$*

$$\Omega_t = \{x \in \mathbb{R}^n : s(x) > t\} \qquad \textit{is an open set.}$$

In the following, we will only use bounded lower semi-continuous functions and we define

$$s_{min} := \inf_{x\in\mathbb{R}^n} s(x) \le s(x) \le \sup_{x\in\mathbb{R}^n} s(x) =: s_{max} \; . \tag{6.3}$$

For $x \in \mathbb{R}^n$ and $K \in \mathbb{N}$ we put

$$s_{K,x} := \inf_{|x-y|\le 2^{-K+2}} s(y) \; . \tag{6.4}$$

By $B^s_p(\mathbb{R}^n)$ we mean the usual Besov spaces $B^s_{pq}(\mathbb{R}^n)$ (this is also $B^{s,mloc}_{pq}(\mathbb{R}^n, \boldsymbol{w})$ with $w_j(x) \equiv 1$) with $p = q$ and the restriction spaces $B^s_p(\Omega)$ are defined for an open $\Omega \subset \mathbb{R}^n$ by

$$B^s_p(\Omega) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \text{ there exists } g \in B^s_p(\mathbb{R}^n) \text{ with } g|_\Omega = f\} \; ,$$

with

$$\big\| f |\, B^s_p(\Omega)\big\| = \inf_{g|_\Omega = f} \big\| g |\, B^s_p(\mathbb{R}^n)\big\| \; . \tag{6.5}$$

Now, we are ready to give the definition of the spaces of varying smoothness.

Definition 6.6: *Let $1 < p \le \infty$ and let $\mathbb{S}$ be a bounded lower semi-continuous function in $\mathbb{R}^n$ with $s_{min} \ge s_0$. Then*

$$B^{\mathbb{S},s_0}_p(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \big\| f |\, B^{\mathbb{S},s_0}_p(\mathbb{R}^n)\big\| < \infty\} \; ,$$

where

$$\big\| f |\, B^{\mathbb{S},s_0}_p(\mathbb{R}^n)\big\| = \big\| f |\, B^{s_0}_p(\mathbb{R}^n)\big\| + \sup_{x\in\mathbb{R}^n} \sup_{K\in\mathbb{N}} 2^{-K(s_{K,x}-s_0)} \big\| f |\, B^{s_{K,x}}_p(B_{2^{-K}}(x))\big\| \; . \tag{6.6}$$

Let us describe how this definition can be understood. The first term checks the global smoothness of a given function f, whereas the supremum term concerns local improvements by the following procedure. For a fixed point $x \in \mathbb{R}^n$ we consider a ball centered in x with radius 2^{-K} and ask if f belongs to the Besov space with smoothness

$s_{K,x} \geq s_0$ in this ball. Now we increase K and therefore shrink the ball around x and ask the same question again with respect to a possibly higher degree of smoothness. We continue this procedure for all K, then all x, and finally check if the supremum over all these norms multiplied by the weight factor $2^{-K(s_{K,x}-s_0)}$ is finite.
To prove the connection of $B_p^{\mathbb{S},s_0}(\mathbb{R}^n)$ to $B_p^{s,s'}(\mathbb{R}^n, U)$ we use the n-dimensional Daubechies wavelets from the last chapter with $k \in \mathbb{N}_0$ large enough and $\operatorname{supp} \psi_M, \operatorname{supp} \psi_F \subset B_{2^J}(0)$. Further, we need a wavelet decomposition theorem of the spaces of varying smoothness. Therefore, we have to introduce some necessary notation. For $x_0 \in \mathbb{R}$ and $K \in \mathbb{N}$ with $K \geq J$ we define

$$f_{K,x_0} = \sum_{\nu,G,m}^{K,x_0} \lambda_{Gm}^{\nu} \Psi_{Gm}^{\nu} \ ,$$

where the summation is restricted to all $\nu > J + K$ and $m \in \mathbb{Z}^n$ with

$$B_{2^{-K+1}}(x_0) \cap B_{2^{-\nu}}(2^{-\nu} m) \neq \emptyset \ .$$

This condition ensures that only those coefficients are taken into account, which depend on information about the function f at most in the ball with radius 2^{-K+2} centered at x_0. The corresponding sequence space norm is given by

$$\left\| \lambda(f) | \, b_p^s \right\|^{K,x_0} = \left\| \lambda(f_{K,x_0}) | \, b_p^s \right\| = \left(\sum_{\nu,G,m}^{K,x_0} 2^{\nu(s-n/p)p} |\lambda_{Gm}^{\nu}|^p \right)^{1/p} \ .$$

The usual Besov sequence spaces corresponding to (2.12) for $0 < p, q \leq \infty$ and $s \in \mathbb{R}^n$ are given by the norms

$$\left\| \lambda | \, b_{pq}^s \right\| = \left(\sum_{\nu=0}^{\infty} 2^{\nu(s-n/p)} \sum_{G \in G^{\nu}} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{Gm}^{\nu}|^p \right)^{q/p} \right)^{1/q} \ . \qquad (6.7)$$

We recall Theorem 5.3 from [Schn07].

Theorem 6.7: *Let $1 < p \leq \infty$, $s_0 < 0$ and $\mathbb{S}$ be a bounded lower semi-continuous function in $\mathbb{R}^n$. Then there are two constants $c_1, c_2 > 0$ such that for all $f \in \mathcal{S}'(\mathbb{R}^n)$ we have*

$$\begin{aligned} c_1 \left\| \lambda(f) | \, b_p^{s_0} \right\| + c_1 \sup_{K \geq J, x \in \mathbb{R}^n} \ & 2^{-K(s_{K,x}-s_0)} \left\| \lambda(f) | \, b_p^{s_{K,x}} \right\|^{K+2,x} \\ & \leq \left\| f | \, B_p^{\mathbb{S},s_0}(\mathbb{R}^n) \right\| \\ \leq c_2 \left\| \lambda(f) | \, b_p^{s_0} \right\| + c_2 \sup_{K \geq J, x \in \mathbb{R}^n} \ & 2^{-K(s_{K,x}-s_0)} \left\| \lambda(f) | \, b_p^{s_{K,x}} \right\|^{K,x} \ . \end{aligned}$$

Finally, we are able to state the first theorem for $B_p^{s,s'}(\mathbb{R}^n, U)$. In the special case of $B_p^{s,s'}(\mathbb{R}^n, x_0)$ it is presented in [Schn07, Theorem 5.9/5.12].

Theorem 6.8: *Let $s' \geq 0$, $1 < p \leq \infty$, $U \subset \mathbb{R}^n$ compact and $f \in B_p^{s,s'}(\mathbb{R}^n, U)$. Then*

$$f \in B_p^{\mathbb{S},s_0}(\mathbb{R}^n) \quad \textit{with } s_0 < 0 \textit{ and } \quad s(x) \leq \begin{cases} s & , x \in U \\ s + s' & , \textit{otherwise} \end{cases}$$

a bounded lower semi-continuous function in $\mathbb{R}^n$ with $s_{min} \geq s_0$.

First of all, let us show that the limiting function

$$s(x) = \begin{cases} s & , x \in U \\ s + s' & , \text{otherwise} \end{cases}$$

is a lower semi-continuous function with respect to Definition 6.5. Since $s + s' \geq s$ we have to differ three cases:

- $-\infty < t < s$, then we have $\Omega_t = \mathbb{R}^n$;
- $s \leq t < s + s'$, then we have $\Omega_t = \mathbb{R}^n \setminus U$;
- $s + s' \leq t < \infty$, then we have $\Omega_t = \emptyset$.

In all three cases the sets Ω_t are open.

Proof: Suppose that f belongs to $B_p^{s,s'}(\mathbb{R}^n, U)$. We obtain from the wavelet decomposition (Corollary 5.33) of $B_p^{s,s'}(\mathbb{R}^n, U)$

$$\left\| \lambda(f) | \, \tilde{b}_p^{s,s'}(U) \right\| = \left(\sum_{\nu,G,m} 2^{\nu(s-n/p)p} |\lambda_{Gm}^\nu|^p (1 + 2^\nu \operatorname{dist}(2^{-\nu} m, U))^{s'p} \right)^{1/p} < \infty \,. \tag{6.8}$$

We want to show that there exists a constant $c > 0$ such that

$$\left\| \lambda(f) | \, b_p^{s_0} \right\| \leq c \left\| \lambda(f) | \, \tilde{b}_p^{s,s'}(U) \right\| \,, \tag{6.9}$$

$$\left\| \lambda(f) | \, b_p^{s_{K,x}} \right\|^{K,x} \leq c 2^{K(s_{K,x} - s_0)} \left\| \lambda(f) | \, \tilde{b}_p^{s,s'}(U) \right\| \,. \tag{6.10}$$

Then we get from Theorem 6.7 that $f \in B_p^{\mathbb{S},s_0}(\mathbb{R}^n)$. The first inequality, (6.9), is easy because of $s_0 \leq s_{min} \leq s$ we obtain

$$\begin{aligned} \left\| \lambda(f) | \, b_p^{s_0} \right\| \leq \left\| \lambda(f) | \, b_p^{s} \right\| &= \left(\sum_{\nu,G,m} 2^{\nu(s-n/p)p} |\lambda_{Gm}^\nu|^p \right)^{1/p} \\ &\leq \left(\sum_{\nu,G,m} 2^{\nu(s-n/p)p} |\lambda_{Gm}^\nu|^p (1 + 2^\nu \operatorname{dist}(2^{-\nu} m, U))^{s'p} \right)^{1/p} = \left\| \lambda(f) | \, \tilde{b}_p^{s,s'}(U) \right\| \,, \end{aligned}$$

where the last inequality comes from $s' \geq 0$. For the second estimate (6.10) we have to distinguish two cases.
<u>First case:</u> We treat all $x \in \mathbb{R}^n$ with $K \geq J$ and $\operatorname{dist}(2^{-\nu}m, U) \leq 2^{-K+2}$. Then we know that $s_{K,x} \leq s$ and we get

$$\begin{aligned}
\left\| \lambda(f) | \, b_p^{s_{K,x}} \right\|^{k,x} \leq \left\| \lambda(f) | \, b_p^{s} \right\|^{K,x} &= \left(\sum_{\nu,G,m}^{K,x} 2^{\nu(s-n/p)p} |\lambda_{Gm}^{\nu}|^p \right)^{1/p} \\
&\leq \left(\sum_{\nu,G,m} 2^{\nu(s-n/p)p} |\lambda_{Gm}^{\nu}|^p (1 + 2^{\nu} \operatorname{dist}(2^{-\nu}m, U))^{s'p} \right)^{1/p} \\
&\leq 2^{K(s_{K,x}-s_0)} \left\| \lambda(f) | \, \tilde{b}_p^{s,s'}(U) \right\| ,
\end{aligned}$$

where we used $s' \geq 0$ again.
<u>Second case:</u> We treat all $x \in \mathbb{R}^n$ with $K \geq J$ and $\operatorname{dist}(2^{-\nu}m, U) > 2^{-K+2}$. Then we have $s_{K,x} \leq s+s'$ and because of $B_{2^{-K+1}}(x) \cap B_{2^{-\nu}}(2^{-\nu}m) \neq \emptyset$ we get $|x - 2^{-\nu}m| \leq 2^{-K+1} + 2^{-\nu}$. This gives us

$$\begin{aligned}
1 + 2^{\nu} \operatorname{dist}(2^{-\nu}m, U) &= 1 + 2^{\nu} \inf_{y \in U} |2^{-\nu}m - y| \\
&\geq 1 + 2^{\nu} \operatorname{dist}(x, U) - 2^{\nu} |2^{-\nu}m - x| \geq 2^{\nu-K+1} \; . \qquad (6.11)
\end{aligned}$$

Finally, we can estimate with (6.11) and $s_{K,x} \leq s + s'$

$$\begin{aligned}
\left\| \lambda(f) | \, b_p^{s_{K,x}} \right\|^{k,x} &= \left(\sum_{\nu,G,m}^{K,x} 2^{\nu(s_{K,x}-n/p)p} |\lambda_{Gm}^{\nu}|^p \frac{(1 + 2^{\nu} \operatorname{dist}(2^{-\nu}m, U))^{s'p}}{(1 + 2^{\nu} \operatorname{dist}(2^{-\nu}m, U))^{s'p}} \right)^{1/p} \\
&\leq 2^{K(s_{K,x}-n/p)} \left(\sum_{\nu,G,m}^{K,x} 2^{(\nu-K)(s_{K,x}-s'-n/p)p} |\lambda_{Gm}^{\nu}|^p (1 + 2^{\nu} \operatorname{dist}(2^{-\nu}m, U))^{s'p} \right)^{1/p} \\
&\leq 2^{K(s_{K,x}-n/p)} 2^{-K(s-n/p)} \left(\sum_{\nu,G,m}^{K,x} 2^{\nu(s-n/p)p} |\lambda_{Gm}^{\nu}|^p (1 + 2^{\nu} \operatorname{dist}(2^{-\nu}m, U))^{s'p} \right)^{1/p} \\
&\leq 2^{K(s_{K,x}-s)} \left\| \lambda(f) | \, \tilde{b}_p^{s,s'}(U) \right\| ,
\end{aligned}$$

and this finishes the proof. □

In the next theorem we would like to show the reverse statement to Theorem 6.8. This means that the two involved spaces are equal. This is not possible because for coefficients $\lambda_{Gm}^{\nu}(f)$ far away from U condition (6.8) is to strong.
Anyway, if we reduce the statements to local versions of $B_p^{s,s'}(\mathbb{R}^n, U)$ then a reverse statement is possible. We say a function f belongs to $B_p^{s,s'}(U)^{loc}$ if there exists a neighborhood $V \supset U$ and a function $h \in B_p^{s,s'}(\mathbb{R}^n, U)$ with $f = h$ on V (see also the next chapter). Now, we can formulate the theorem which is similar to Theorem 5.10 in [Schn07] for the $C_{x_0}^{s,s'}(\mathbb{R}^n)^{loc}$ case.

Theorem 6.9: *Let $s<0$, $s'\geq 0$, $U\subset\mathbb{R}^n$ compact, $1<p<\infty$ and $f\in B_p^{\mathbb{S},s}(\mathbb{R}^n)$ with*

$$s(x)=\begin{cases} s & ,x\in U\\ s+s' & ,otherwise\ .\end{cases}$$

Then $f\in B_p^{\sigma,\sigma'}(U)^{loc}$ with $\sigma<s$ and $\sigma'\in\mathbb{R}$ with $\sigma+\sigma'<s+s'-\frac{n}{p}$.

Proof: At first, we need some preparation. From a pointwise multiplier theorem for $B_p^{\mathbb{S},s}(\mathbb{R}^n)$ (Theorem 3.8 in [Schn07]) we know that it is sufficient to prove $\varphi f\in B_p^{s,s'}(\mathbb{R}^n,U)$ for a $\varphi\in C^\infty(\mathbb{R}^n)$ with

$$\varphi(x)=\begin{cases} 1 & \text{, for } \operatorname{dist}(x,U)\leq 1/2\\ 0 & \text{, for } \operatorname{dist}(x,U)\geq 1\ .\end{cases} \tag{6.12}$$

Hence, it is even enough to show $g\in B_p^{s,s'}(\mathbb{R}^n,U)$ for every function $g\in B_p^{\mathbb{S},s}(\mathbb{R}^n)$ with $\operatorname{supp} g\subset U_1=\{x\in\mathbb{R}^n:\operatorname{dist}(x,U)\leq 1\}$.
That means that the coefficients $\lambda^\nu_{Gm}(g)=0$ for $\operatorname{dist}(2^{-\nu}m,U)>1+2^{J-\nu}$. Since $g\in B_p^{\mathbb{S},s}(\mathbb{R}^n)$, we have from Theorem 6.7

$$\left\|\lambda(g)|\,b_p^s\right\|\leq c \quad \text{and} \tag{6.13}$$
$$\left\|\lambda(g)|\,b_p^{s_{K,x}}\right\|^{K+2,x}\leq c2^{K(s_{K,x}-s)} \quad \text{for all } K\geq J \text{ and } x\in\mathbb{R}^n. \tag{6.14}$$

We have to show that

$$\left\|\lambda(g)|\,b_p^{\sigma,\sigma'}U\right\|^p=\sum_{\nu,G,m}2^{\nu(s-n/p)p}|\lambda^\nu_{Gm}(g)|^p(1+2^\nu\operatorname{dist}(2^{-\nu}m,U))^{\sigma'p} \tag{6.15}$$
$$=\sum{}^1+\sum{}^2+\sum{}^3+\sum{}^0<\infty\ . \tag{6.16}$$

In the following we consider $\sigma'\geq 0$. The part with $\sigma'<0$ can be incorporated afterwards. The sum with the zero considers all coefficients with $\operatorname{dist}(2^{-\nu}m,U)>1+2^{J-\nu}$ and this is equal 0. From now on all coefficients regarded satisfy $\operatorname{dist}(2^{-\nu}m,U)\leq 1+2^{J-\nu}$.
The first sum considers the coefficients with $\nu<\nu_0$ for some $\nu_0\in\mathbb{N}$ and we have

$$\left(\sum{}^1\right)^{1/p}\leq c_{\nu_0}\left\|\lambda|\,b_p^s\right\|$$

because of $(1+2^\nu\operatorname{dist}(2^{-\nu}m,U))^{\sigma'}\leq(1+2^{\nu_0}+2^J)^{\sigma'}=c_{\nu_0}$. We fix $\nu_0=J+4$ in the sequel.
The second sum obtains the coefficients, for which we can find an $i\in\{\nu_0,\dots,\nu\}$ with

$$2^{-\nu+i}<\operatorname{dist}(2^{-\nu}m,U)\leq 2^{-\nu+i+1}\ . \tag{6.17}$$

We denote $\Gamma_{i\nu}=\{m\in\mathbb{Z}^n:2^{-\nu+i}<\operatorname{dist}(2^{-\nu}m,U)\leq 2^{-\nu+i+1}\}$. Then we have for $x_1=2^{-\nu}m$ and $K_1=\nu-i+3$

$$B_{2^{-K_1-1}}(x_1)\cap B_{2^{-\nu}}(2^{-\nu}m)\neq\emptyset \quad\text{and}\quad \operatorname{dist}(x_1,U)>2^{-j+i}>2^{-K_1+2}\ . \tag{6.18}$$

We obtain $s_{K_1,x_1} = s + s'$ and we derive for fixed ν and m by (6.14)

$$\begin{aligned} 2^{\nu(s+s'-n/p)p}|\lambda^{\nu}_{Gm}|^p &\leq \sum_{j,G,m}{}^{K_1+2,x_1} 2^{\nu(s_{K_1,x_1}-n/p)p}|\lambda^{\nu}_{Gm}|^p \\ &= \left(\left\|\lambda|\, b_p^{s_{K,x}}\right\|^{K_1+2,x_1}\right)^p \leq c2^{K_1(s_{K_1,x_1}-s)p} \ . \end{aligned} \tag{6.19}$$

Hence,

$$\begin{aligned} \left(\sum{}^2\right)^{1/p} &= \left(\sum_{\nu=\nu_0}^{\infty}\sum_{G\in G^\nu}\sum_{i=\nu_0}^{\nu}\sum_{m\in\Gamma_{i\nu}} 2^{\nu(\sigma-n/p)p}|\lambda^{\nu}_{Gm}|^p(1+2^\nu\operatorname{dist}(2^{-\nu}m,U))^{\sigma'p}\right)^{1/p} \\ &\leq c\left(\sum_{\nu=\nu_0}^{\infty}\sum_{G\in G^\nu}\sum_{i=\nu_0}^{\nu}\sum_{m\in\Gamma_{i\nu}} 2^{\nu(\sigma-s-s')p}2^{K_1(s_{K_1,x_1}-s)p}(1+2^{i+1})^{\sigma'p}\right)^{1/p} . \end{aligned}$$

The sum over G^ν is finite and $|\Gamma_{i\nu}| \sim 2^{ni}$ and we get

$$\left(\sum{}^2\right)^{1/p} \leq c'\left(\sum_{\nu=\nu_0}^{\infty} 2^{\nu(\sigma-s)p}\sum_{i=\nu_0}^{\nu}2^{i(\sigma'-s'+n/p)p}\right)^{1/p} < \infty \quad \text{for } \sigma+\sigma' < s+s'-n/p.$$

Finally, the third sum contains the coefficients with $\operatorname{dist}(2^{-\nu}m,U) \leq 2^{-\nu+\nu_0}$. Again we define $K_2 = \nu-\nu_0+2$ and $x_2 = 2^{-\nu}m$ and get

$$B_{2^{-K_2-1}}(x_2)\cap B_{2^{-\nu}}(2^{-\nu}m) \neq \emptyset \quad \text{and} \quad \operatorname{dist}(x_2,U) \leq 2^{-K_2+2} \ .$$

We obtain $s_{K_2,x_2} = s$ and we derive for fixed ν and m by (6.14)

$$\begin{aligned} 2^{\nu(s-n/p)p}|\lambda^{\nu}_{Gm}|^p &\leq \sum_{j,G,m}{}^{K_2+2,x_2} 2^{\nu(s_{K_2,x_2}-n/p)p}|\lambda^{\nu}_{Gm}|^p \\ &= \left(\left\|\lambda|\, b_p^{s_{K,x}}\right\|^{K_2+2,x_2}\right)^p \leq c2^{K_2(s_{K_2,x_2}-s)p} = c \ . \end{aligned} \tag{6.20}$$

Consequently, we can estimate with (6.20)

$$\begin{aligned} \left(\sum{}^3\right)^{1/p} &= \left(\sum_{\nu=\nu_0}^{\infty}\sum_{G\in G^\nu}\sum_{\operatorname{dist}(2^{-\nu}m,U)\leq 2^{-\nu+\nu_0}} 2^{\nu(\sigma-n/p)p}|\lambda^{\nu}_{Gm}|^p(1+2^\nu\operatorname{dist}(2^{-\nu}m,U))^{\sigma'p}\right)^{1/p} \\ &\leq c\left(\sum_{\nu=\nu_0}^{\infty}\sum_{G\in G^\nu}\sum_{\operatorname{dist}(2^{-\nu}m,U)\leq 2^{-\nu+\nu_0}} 2^{\nu(\sigma-s)p}2^{K_2(s_{K_2,x_2}-s)p}(1+2^{\nu_0})^{\sigma'p}\right)^{1/p} \\ &\leq c'\left(\sum_{\nu=\nu_0}^{\infty}2^{\nu(\sigma-s)p}\right)^{1/p} < \infty \ , \end{aligned}$$

because $|G^\nu| = 2^n-1$, the cardinality of $m\in\mathbb{Z}^n$ satisfying $\operatorname{dist}(2^{-\nu}m,U) \leq 2^{-\nu+\nu_0}$ is equal to $2^{\nu_0 n}$ and $\sigma < s$. The case $\sigma' < 0$ follows the same reasoning and this remark finishes the proof. $\square$

With the same proof a better result holds in the case $p = \infty$. In the case $U = \{x_0\}$ it is already known, see [Schn07, Theorem 5.10].

Theorem 6.10: *Let $s < 0$, $s' \geq 0$, $U \subset \mathbb{R}^n$ compact and $f \in B_{\infty}^{\mathbb{S},s}(\mathbb{R}^n)$ with*

$$s(x) = \begin{cases} s & ,x \in U \\ s + s' & ,otherwise \ . \end{cases}$$

Then $f \in B_{\infty}^{s,s'}(U)^{loc}$.

Remark 6.11: If one defines the spaces of varying smoothness by the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that the coefficients $\lambda_{Gm}^{\nu}(f)$ of the wavelet decomposition satisfy

$$\left\| \lambda(f) | \, b_p^{s_0} \right\| + \sup_{K \geq J, x \in \mathbb{R}^n} 2^{-K(s_{K,x} - s_0)} \left\| \lambda(f) | \, b_p^{s_{K,x}} \right\|^{K,x} < \infty \ ,$$

then it is possible to extend the definition of the spaces to $0 < p \leq \infty$. Since we only used the wavelet characterization of the spaces and nowhere that $p > 1$, we can obtain the above theorems also for the $p \leq 1$ case.

6.3 Sharp embeddings

In this section we want to apply the wavelet characterization to get sharp versions of the embedding theorems from Section 2.5.1. We want to show the method only for the embedding (2.44), the others can be treated in the same manner.

Theorem 6.12: *Let $0 < q \leq \infty$, $0 < p_1, p_2 \leq \infty$, $s_1, s_2 \in \mathbb{R}$ and $\boldsymbol{w}, \boldsymbol{\varrho} \in \mathcal{W}_{\alpha_1,\alpha_2}^{\alpha}$. Then*

$$B_{p_1 q}^{s_1,mloc}(\mathbb{R}^n, \boldsymbol{\varrho}) \hookrightarrow B_{p_2 q}^{s_2,mloc}(\mathbb{R}^n, \boldsymbol{w})$$

holds if, and only if,

$$0 < p_1 \leq p_2 \leq \infty \ and \ \frac{w_\nu(x)}{\varrho_\nu(x)} \leq c 2^{\nu\left(s_1 - s_2 - n(\frac{1}{p_1} - \frac{1}{p_2})\right)} \ . \tag{6.21}$$

Proof: First of all, we know that the reverse direction of the theorem is proved in Section 2.5.1. Nevertheless, we like to mention that it is much more easy if one uses the wavelet characterization. Therefore, let $\{\Psi_{Gm}^{\nu}\}$ be Daubechies wavelets with sufficiently large $k \in \mathbb{N}_0$. We have to prove that there is a constant $c > 0$ such that

$$\left\| \lambda | \, \tilde{b}_{p_2,q}^{s_2,mloc}(\boldsymbol{w}) \right\| \leq c \left\| \lambda | \, \tilde{b}_{p_1,q}^{s_1,mloc}(\boldsymbol{\varrho}) \right\| \quad \text{holds for all } \lambda \in \tilde{b}_{p_1,q}^{s_1,mloc}(\boldsymbol{\varrho}). \tag{6.22}$$

But this is an easy consequence of $\ell_{p_1} \hookrightarrow \ell_{p_2}$ and condition (6.21).
For the necessity of the condition (6.21) we only consider $p_1 = p_2 = p$. The condition $p_1 \leq p_2$ follows easily by taking sequences with $\lambda_{Gm}^{\nu} = 0$ for $\nu \geq 1$. Then it is easy to construct a contradiction with the help of the harmonic series.
Hence, in the case $p_1 = p_2 = p$ we like to show that (6.22) deduces that there exists a

$c > 0$ with $\frac{w_\nu(x)}{\varrho_\nu(x)} \leq c2^{\nu(s_1-s_2)}$ for all $\nu \in \mathbb{N}_0$ and all $x \in \mathbb{R}^n$. The proof is indirect; we assume that for every $\nu \in \mathbb{N}_0$ there exists a $m_\nu \in \mathbb{Z}^n$ with

$$w_\nu(2^{-\nu}m_\nu) \geq c_\nu 2^{\nu(s_1-s_2)}\varrho_\nu(2^{-\nu}m_\nu) \quad \text{and } c_\nu \text{ is unbounded.} \tag{6.23}$$

Now, for every $\nu \in \mathbb{N}_0$ we define a sequence $\lambda_\nu \in \tilde{b}^{s_1,mloc}_{p,q}(\boldsymbol{\varrho})$ which only has one non zero coefficient $\lambda^\nu_{Gm_\nu} = 2^{-\nu(s_1-n/p)}\varrho_\nu^{-1}(2^{-\nu}m_\nu)$. We have, $\left\|\lambda_\nu|\, \tilde{b}^{s_1,mloc}_{p,q}(\boldsymbol{\varrho})\right\| = 1$ and from (6.23) we obtain $\left\|\lambda_\nu|\, \tilde{b}^{s_2,mloc}_{p,q}(\boldsymbol{w})\right\| \geq c_\nu$ which is unbounded and therefore contradicts (6.22). $\square$

Finally, we give a sharp version of one case of Theorem 2.33 for the 2-microlocal spaces $B^{s,s'}_{pq}(\mathbb{R}^n, U)$. It is a special case of the last theorem and Remark 2.34.

Theorem 6.13: *Let $U \subset \mathbb{R}^n$ be bounded. Then*

$$B^{s,s'}_{pq}(\mathbb{R}^n, U) \hookrightarrow B^{t,t'}_{pq}(\mathbb{R}^n, U)$$

holds if, and only if,

$$s' \geq t' \text{ and } s \geq t \ .$$

6.4 Delta distribution

In this section we give exact conditions on the parameters such that the delta-distribution $\delta(\varphi) = \varphi(0)$ belongs to $B^{s,s'}_{pq}(\mathbb{R}^n, U)$.

Theorem 6.14: *Let $U \subset \mathbb{R}^n$ be bounded, $s, s' \in \mathbb{R}$ and $0 < p \leq \infty$.*

(i) Let $\operatorname{dist}(0, U) = 0$, then we have

$$\delta \in B^{s,s'}_{pq}(\mathbb{R}^n, U) \quad \text{if, and only if,} \quad s < \frac{n}{p} - n \quad \text{for } 0 < q < \infty \tag{6.24}$$

and

$$\delta \in B^{s,s'}_{p\infty}(\mathbb{R}^n, U) \quad \text{if, and only if,} \quad s \leq \frac{n}{p} - n \ . \tag{6.25}$$

(ii) Let $\operatorname{dist}(0, U) = \eta > 0$, then we have

$$\delta \in B^{s,s'}_{pq}(\mathbb{R}^n, U) \quad \text{if, and only if,} \quad s + s' < \frac{n}{p} - n \quad \text{for } 0 < q < \infty \tag{6.26}$$

and

$$\delta \in B^{s,s'}_{p\infty}(\mathbb{R}^n, U) \quad \text{if, and only if,} \quad s + s' \leq \frac{n}{p} - n \ . \tag{6.27}$$

Proof: We only proof the cases, where $0 < q < \infty$ because for $q = \infty$ the proofs are analogous. As wavelet basis we take again the Daubechies wavelets with $k \in \mathbb{N}_0$ sufficiently large and $\operatorname{supp} \psi_M, \operatorname{supp} \psi_F \subset B_{2^J}(0)$. Then we have

$$\lambda^\nu_{Gm}(\delta) = 2^{\nu n/2} \langle \delta, \Psi^\nu_{Gm} \rangle = 2^{\nu n} \prod_{r=1}^n \psi_{G_r}(0 - m_r) \quad \text{where } G = (G_1, \ldots, G_n) \in G^\nu.$$

We obtain from the support property of the wavelets that $\lambda^\nu_{Gm}(\delta) = 0$ for $m \notin Q_J = [-2^J, 2^J]^n$. Now, we can estimate from $\operatorname{dist}(0, U) = 0$ that $1 \leq (1 + 2^\nu \operatorname{dist}(2^{-\nu} m, U)) \leq (1 + 2^\nu \operatorname{dist}(0, U) + |m|) \leq c_J$ and that leads us for all $s' \in \mathbb{R}$ to

$$\begin{aligned}
\left\| \lambda(\delta) | \, b^{s,s'}_{pq}(U) \right\| &= \left(\sum_{\nu=0}^\infty 2^{\nu(s-n/p)q} \sum_{G \in G^\nu} \left(\sum_{m \in \mathbb{Z}^n} |\lambda^\nu_{Gm}|^p (1 + 2^\nu \operatorname{dist}(2^{-\nu} m, U))^{s'p} \right)^{q/p} \right)^{1/p} \\
&= \left(\sum_{\nu=0}^\infty 2^{\nu(s-n/p)q} \sum_{G \in G^\nu} \left(\sum_{m \in \mathbb{Z}^n} 2^{\nu n/p} |\Psi^\nu_{Gm}(0)|^p (1 + 2^\nu \operatorname{dist}(2^{-\nu} m, U))^{s'p} \right)^{q/p} \right)^{1/p} \\
&\leq c \left(\sum_{\nu=0}^\infty 2^{\nu(s-n/p+n)p} \right)^{1/p} < \infty \quad \text{for } s < \frac{n}{p} - n.
\end{aligned}$$

If $\operatorname{dist}(0, U) = \eta > 0$, then we have to differ two cases. The first one, $s' \geq 0$, uses the calculation above and $(1 + 2^\nu \operatorname{dist}(0, U))^{s'} \leq 2^{\nu s'} (1 + \eta + |m|)^{s'} = c 2^{\nu s'}$. The norm is less than infinity for $s + s' < \frac{n}{p} - n$. In the second case, $s' < 0$, we have to work more careful. We know that $\lambda^\nu_{Gm}(\delta) = 0$ for $m \in \mathbb{Z}^n \setminus Q_J$. We choose $\nu_0 \in \mathbb{N}_0$ large enough, with $|2^{-\nu} m| \leq \eta/2$ for $\nu > \nu_0$ and $m \in Q_J$. Then we derive for these ν and m that $\operatorname{dist}(2^{-\nu} m, U) \geq \eta/2$, hence

$$\begin{aligned}
\left\| \lambda(\delta) | \, b^{s,s'}_{pq}(U) \right\| &= \left(\sum_{\nu=0}^\infty 2^{\nu(s-n/p)q} \sum_{G \in G^\nu} \left(\sum_{m \in \mathbb{Z}^n} |\lambda^\nu_{Gm}|^p (1 + 2^\nu \operatorname{dist}(2^{-\nu} m, U))^{s'p} \right)^{q/p} \right)^{1/p} \\
&\leq c \left(\sum_{\nu=0}^{\nu_0} 2^{\nu(s-n/p+n)q} \left(\sum_{m \in Q_J} (1 + 2^\nu \operatorname{dist}(2^{-\nu} m, U))^{s'p} \right)^{q/p} \right)^{1/p} \\
&\qquad + c \left(\sum_{\nu=\nu_0+1}^\infty 2^{\nu(s-n/p+n)q} \left(\sum_{m \in Q_J} (1 + 2^\nu \operatorname{dist}(2^{-\nu} m, U))^{s'p} \right)^{q/p} \right)^{1/p} \\
&\leq c' \left(\sum_{\nu=0}^{\nu_0} 2^{\nu(s-n/p+n)p} \right)^{1/p} + c \left(\sum_{\nu=\nu_0+1}^\infty 2^{\nu(s-n/p+n)q} \left(\sum_{m \in Q_J} (1 + 2^\nu \eta/2)^{s'p} \right)^{q/p} \right)^{1/p} \\
&\leq c'' + c' \left(\sum_{\nu=\nu_0+1}^\infty 2^{\nu(s+s'-n/p+n)p} \right)^{1/p} < \infty \quad \text{for } s + s' < \frac{n}{p} - n.
\end{aligned}$$

Finally, we want to prove that the conditions on the parameters are sharp. Therefore we take some wavelet basis $\{\Psi^{\nu}_{Gm}\}$ which is admissible with respect to our wavelet characterization. The only thing we need is that $|\psi_M(0)| = c > 0$. Then we conclude indirectly; assume in the first case that $s \geq n/p - n$, then taking only $m = 0$ and $G = (M, \ldots, M)$ and $\operatorname{dist}(0, U) = 0$

$$\left\| \lambda(\delta) | \, b^{s,s'}_{pq}(U) \right\| \geq c \left(\sum_{\nu=0}^{\infty} 2^{\nu(s - \frac{n}{p} + n)q} (1 + 2^{\nu} \operatorname{dist}(0, U))^{s'q} \right)^{1/q}$$

$$= c \left(\sum_{\nu=0}^{\infty} 2^{\nu(s - \frac{n}{p} + n)q} \right)^{1/q} = \infty \ ,$$

which contradicts $\delta \in B^{s,s'}_{pq}(\mathbb{R}^n, U)$. In the case $\operatorname{dist}(0, U) = \eta > 0$ we make the same calculation and use $(1 + 2^{\nu} \operatorname{dist}(0, U))^{s'} \geq c 2^{\nu s'}$ for all $s' \in \mathbb{R}$. $\square$

7 The local spaces $\mathbf{B^{s,s'}_{pq}(U)^{loc}}$

This chapter is devoted to the study of the local spaces $B^{s,s'}_{pq}(U)^{loc}$. They are an appropriate instrument for measuring local regularity of functions and they were treated intensively by Jaffard, Meyer, Seuret, Levy-Véhel and many others ([JaMey96], [LVSeu04]). We would like to point out some connections to the known case, $p = q = \infty$ and $U = \{x_0\}$, and give first results.

7.1 Definition and wavelet characterization

In this section we define the local version of $B^{s,s'}_{pq}(\mathbb{R}^n, U)$ for compact $U \subset \mathbb{R}^n$.

Definition 7.1: *Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $U \subset \mathbb{R}^n$ compact, then f belongs to the local space $B^{s,s'}_{pq}(U)^{loc}$ if there exists an open neighborhood $U \subset V_0$ and $g \in B^{s,s'}_{pq}(\mathbb{R}^n, U)$ globally such that $f = g$ on V_0.*

From Theorem 4.3 we obtain the following.

Lemma 7.2: *Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $U \subset \mathbb{R}^n$ compact. Then $f \in B^{s,s'}_{pq}(U)^{loc}$ if, and only if, there exists an open neighborhood $U \subset V_0$ and $\varphi \in C^\infty_0(\mathbb{R}^n)$ with $\varphi(x) = 1$ on V_0 and $\varphi f \in B^{s,s'}_{pq}(\mathbb{R}^n, U)$.*

The most important characterization of the local spaces is due to the wavelet characterization. Therefore we have to modify the notation of the wavelet characterization of Section 5.3. We overtake the notation from [Tri06, 4.2.1]. Let $\psi_M, \psi_F \in C^k(\mathbb{R}^n)$ be the Daubechies wavelets with sufficiently large $k \in \mathbb{N}_0$ and $\operatorname{supp}\psi_M, \operatorname{supp}\psi_F \subset B_{2^J}(0)$. Let $l \in \mathbb{N}_0$ then

$$G = G^{l,l} = \{F, M\}^n \qquad \text{and} \qquad G^{\nu,l} = \{F, M\}^{n*} \quad \text{for } \nu > l.$$

We obtain for fixed $l \in \mathbb{N}_0$ that $\{\Psi^{\nu,l}_{Gm} : \nu \geq l,\ G \in G^{\nu,l} \text{ and } m \in \mathbb{Z}^n\}$ is an orthonormal basis of $L_2(\mathbb{R}^n)$ with

$$\Psi^{\nu,l}_{Gm}(x) = 2^{\nu\frac{n}{2}} \prod_{r=1}^{n} \psi_{G_r}(2^\nu x_r - m_r) \quad \text{where } G = (G_1, \ldots, G_n) \in G^{\nu,l}.$$

We have to adapt our sequence spaces to the new situation. We say a sequence of complex-valued numbers $\{\lambda^{\nu,l}_{Gm}\}$ belongs to $\tilde{b}^{s,s'}_{pq;l}(U)$ if

$$\left\| \lambda | \, \tilde{b}^{s,s'}_{pq;l}(U) \right\| = \left(\sum_{\nu=l}^{\infty} 2^{\nu(s-n/p)q} \sum_{G \in G^{\nu,l}} \left(\sum_{m \in \mathbb{Z}^n} |\lambda^{\nu,l}_{Gm}|^p (1 + 2^\nu \operatorname{dist}(2^{-\nu}m, U))^{s'p} \right)^{q/p} \right)^{1/q} < \infty \,.$$

Now, we can modify the proofs and statements and we get analogous to Corollary 5.33 the following.

Corollary 7.3: *Let $U \subset \mathbb{R}^n$ bounded, $s' \in \mathbb{R}$, $l \in \mathbb{N}_0$ and $w_\nu(x) = (1 + 2^\nu \operatorname{dist}(x, U))^{s'}$. Further, let $s \in \mathbb{R}$, $0 < p, q \leq \infty$ and*

$$k > \max(\sigma_p - s - \min(0, s'), s + \max(0, s')) \tag{7.1}$$

in (5.76) and (5.77). Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then $f \in B^{s,s'}_{pq}(\mathbb{R}^n, U)$ if, and only if, it can be represented as

$$f = \sum_{\nu=l}^{\infty} \sum_{G \in G^{\nu,l}} \sum_{m \in \mathbb{Z}^n} \lambda^{\nu,l}_{Gm} 2^{-\nu\frac{n}{2}} \Psi^{\nu,l}_{Gm} \qquad \text{with } \lambda \in \tilde{b}^{s,s'}_{pq;l}(U) \ , \tag{7.2}$$

with unconditional convergence in $\mathcal{S}'(\mathbb{R}^n)$ and in any $B^{t,t'}_{pq}(\mathbb{R}^n, U)$ with $t < s$ and $t' < s'$. The representation (7.2) is unique,

$$\lambda^{\nu,l}_{Gm} = \lambda^{\nu,l}_{Gm}(f) = 2^{\nu\frac{n}{2}} \left\langle f, \Psi^{\nu,l}_{Gm} \right\rangle \tag{7.3}$$

and

$$I : f \mapsto \{2^{\nu\frac{n}{2}} \left\langle f, \Psi^{\nu,l}_{Gm} \right\rangle\} \tag{7.4}$$

is an isomorphic map from $B^{s,s'}_{pq}(\mathbb{R}^n, U)$ onto $\tilde{b}^{s,s'}_{pq;l}(U)$. Moreover, if in addition $\max(p, q) < \infty$ then $\{\Psi^{\nu,l}_{Gm}\}$ is in unconditional basis in $B^{s,s'}_{pq}(\mathbb{R}^n, U)$.

The advantage of this representation with extra index $l \in \mathbb{N}_0$ is that the size of the support of the wavelets on the zero level $\nu = l$ is $\operatorname{supp} \Psi^{l,l}_{Gm} \subset B_{2^{J-l}}(2^{-l}m)$ and can be minimized in taking large $l \in \mathbb{N}_0$.
We assume in the following that the Daubechies wavelets have enough regularity, which means $k > \max(\sigma_p - s - \min(0, s'), s + \max(0, s'))$, which is in the case $p \geq 1$ equal to $k > \max(|s|, |s + s'|)$.

Theorem 7.4: *Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $U \subset \mathbb{R}^n$ compact. Then f belongs to $B^{s,s'}_{pq}(U)^{loc}$ if, and only if, there exists an $l \in \mathbb{N}_0$ and an $A > 0$ with*

$$\left(\sum_{\nu=l}^{\infty} 2^{\nu(s-n/p)q} \sum_{G \in G^{\nu,l}} \left(\sum_{m \in U_\nu} |\lambda^{\nu,l}_{Gm}(f)|^p (1 + 2^\nu \operatorname{dist}(2^{-\nu}m, U))^{s'p} \right)^{q/p} \right)^{1/q} < \infty \ , \tag{7.5}$$

where

$$U_\nu = \{m \in \mathbb{Z}^n : \operatorname{dist}(2^{-\nu}m, U) \leq A\} \quad \text{and} \quad \lambda^{\nu,l}_{Gm}(f) = 2^{\nu n/2} \left\langle f, \Psi^{\nu,l}_{Gm} \right\rangle \ .$$

Proof: First Step: We have $f \in B^{s,s'}_{pq}(U)^{loc}$, which means that we can find $U \subset V_0 \subset V$ and $\varphi \in C^\infty_0(\mathbb{R}^n)$ with $\varphi(x) = 1$ on V_0, $\operatorname{supp} \varphi \subset V$ and $\varphi f \in B^{s,s'}_{pq}(\mathbb{R}^n, U)$. We can find a number, $-h \in \mathbb{N}_0$ such that $U_{2^h} \subset V_0$, where $U_{2^h} = \{x \in \mathbb{R}^n : \operatorname{dist}(x, U) \leq 2^h\}$. We would like to take these $\Psi^{\nu,l}_{Gm}$ which fulfill

$$\left\langle \varphi f, \Psi^{\nu,l}_{Gm} \right\rangle = \left\langle f, \Psi^{\nu,l}_{Gm} \right\rangle \ , \tag{7.6}$$

which means that $\operatorname{supp}\Psi^{\nu,l}_{Gm} \subset U_{2^h} \subset V_0$. This is fulfilled by $\operatorname{dist}(2^{-j}m, U) \leq 2^h - 2^{J-\nu}$ and to have a positive number on the right hand side we have to demand $\nu > J - h$. Now, we fix $l = J - h + 1$ and $A > 0$ by $A = 2^h - 2^{J-l}$.
From Corollary 7.3 we derive that

$$\left(\sum_{\nu=l}^{\infty} 2^{\nu(s-n/p)q} \sum_{G\in G^{\nu,l}} \left(\sum_{m\in\mathbb{Z}^n} |\lambda^{\nu,l}_{Gm}(\varphi f)|^p (1 + 2^\nu \operatorname{dist}(2^{-\nu}m, U))^{s'p}\right)^{q/p}\right)^{1/q} < \infty$$

and that finally gives us with (7.6) and $U_\nu = \{m \in \mathbb{Z}^n : \operatorname{dist}(2^{-\nu}m, U) \leq A\}$ with $A > 0$ as above

$$\left(\sum_{\nu=l}^{\infty} 2^{\nu(s-n/p)q} \sum_{G\in G^{\nu,l}} \left(\sum_{m\in U_\nu} |\lambda^{\nu,l}_{Gm}(f)|^p (1 + 2^\nu \operatorname{dist}(2^{-\nu}m, U))^{s'p}\right)^{q/p}\right)^{1/q} < \infty \ .$$

Second step: If we have (7.5) for some $l \in \mathbb{N}_0$ and $A > 0$, then we can define

$$\tilde{\lambda}^{\nu,l}_{Gm} = \begin{cases} \lambda^{\nu,l}_{Gm} & \text{, for } m \in U_\nu \\ 0 & \text{, otherwise.} \end{cases}$$

Then $f = \sum_{\nu,G,m} \tilde{\lambda}^{\nu,l}_{Gm} 2^{-\nu\frac{n}{2}} \Psi^{\nu,l}_{Gm}$ belongs by Corollary 7.3 to $B^{s,s'}_{pq}(\mathbb{R}^n, U)$ and that implies $f \in B^{s,s'}_{pq}(U)^{loc}$. $\square$

Remark 7.5: We like to point out that this theorem is similar to [JaMey96, Proposition 1.4] and [LVSeu04, Theorem 1] in the cases $p = q = \infty$, $p = q = 2$ and $U = \{x_0\}$.

7.2 Embeddings

In this section we like to present some embedding theorems for the local spaces. These embeddings are well known in the case $p = q = \infty$ and $U = \{x_0\}$.

Lemma 7.6: *Let $f \in B^{s,s'}_{pq}(U)^{loc}$, then f belongs to $B^{s-\varepsilon,s'+\varepsilon}_{pq}(U)^{loc}$ for every $\varepsilon > 0$.*

Proof: For $f \in B^{s,s'}_{pq}(U)^{loc}$ we find an $l \in \mathbb{N}_0$ and $A > 0$ such that by Theorem 7.4

$$\left(\sum_{\nu=l}^{\infty} 2^{\nu(s-n/p)q} \sum_{G\in G^{\nu,l}} \left(\sum_{m\in U_\nu} |\lambda^{\nu,l}_{Gm}(f)|^p (1 + 2^\nu \operatorname{dist}(2^{-\nu}m, U))^{s'p}\right)^{q/p}\right)^{1/q} < \infty \ ,$$

where $U_\nu = \{m \in \mathbb{Z}^n : \operatorname{dist}(2^{-\nu}m, U) \leq A\}$. Now, we get by $(1 + 2^\nu \operatorname{dist}(2^{-\nu}m, U))^\varepsilon \leq$

$(1+A)^{\varepsilon} 2^{\nu\varepsilon}$ that

$$\left(\sum_{\nu=l}^{\infty} 2^{\nu(s-\varepsilon-n/p)q} \sum_{G\in G^{\nu,l}} \left(\sum_{m\in U_\nu} |\lambda^{\nu,l}_{Gm}(f)|^p (1+2^\nu \operatorname{dist}(2^{-\nu}m,U))^{(s'+\varepsilon)p}\right)^{q/p}\right)^{1/q}$$

$$\leq c\left(\sum_{\nu=l}^{\infty} 2^{\nu(s-\varepsilon-n/p)q} \sum_{G\in G^{\nu,l}} \left(\sum_{m\in U_\nu} |\lambda^{\nu,l}_{Gm}(f)|^p 2^{\nu\varepsilon p} (1+2^\nu \operatorname{dist}(2^{-\nu}m,U))^{s'p}\right)^{q/p}\right)^{1/q}$$

$$\leq \left(\sum_{\nu=l}^{\infty} 2^{\nu(s-n/p)q} \sum_{G\in G^{\nu,l}} \left(\sum_{m\in U_\nu} |\lambda^{\nu,l}_{Gm}(f)|^p (1+2^\nu \operatorname{dist}(2^{-\nu}m,U))^{s'p}\right)^{q/p}\right)^{1/q} < \infty ,$$

and Theorem 7.4 again finishes the proof. □

More generally we can proof the following embedding.

Theorem 7.7:

$$B^{s,s'}_{pq}(U)^{loc} \hookrightarrow B^{t,t'}_{pq}(U)^{loc} \quad \textit{if, and only if,} \quad t \leq s \textit{ and } t+t' \leq s+s' .$$

Proof: The sufficiency of the conditions on the parameter $s, t, s', t' \in \mathbb{R}$ is proved as in the above lemma. To get the necessity we have to be more careful. The embedding is equivalent to that we can find $l \in \mathbb{N}_0$, $A > 0$ and $c > 0$ such that

$$2^{(t-s)\nu} \leq c(1+2^\nu \operatorname{dist}(2^{-\nu}m,U))^{s'-t'} \quad \text{holds for all } \nu \geq l \text{ and } m \in U_\nu. \tag{7.7}$$

We have to distinguish two cases. First, we assume that $s < t$, then for $\nu \geq l$ large enough, we can find $m_\nu \in U_\nu$ with $\operatorname{dist}(2^{-\nu}m_\nu, U) \sim 2^{-\nu}$. This implies that the left hand side of (7.7) is increasing in ν but the right hand side is independent on ν which is a contradiction to (7.7).

In the second case we assume that $t + t' > s + s'$. Then we take for every $\nu \geq l$ an $m_\nu \in U_\nu$ with $\operatorname{dist}(2^{-\nu}m_\nu, U) \sim A$. We can estimate the right hand side of (7.7) by

$$(1+2^\nu \operatorname{dist}(2^{-\nu}m_\nu,U))^{s'-t'} \leq c2^{\nu(s'-t')} \quad \text{where } c > 0 \text{ is independent of } \nu.$$

This gives us the contradiction to (7.7), because there exists no $c > 0$ with $2^{\nu(t-s)} \leq c2^{\nu(s'-t')}$ for all $\nu \geq l$. □

Remark 7.8: This embedding theorem is in contrast to Theorem 6.13 and it is well known in the case of the local $C^{s,s'}_{x_0}(\mathbb{R}^n)$ ([Mey97, Corollary III/3.4]). Moreover, this theorem is the starting point for the definition of the so-called 2-microlocal frontier, see [Mey97, III.5] and [LVSeu04, Chapter 2].

7.3 The 2-microlocal domain

In this subsection we give a generalized approach to define on the basis of the theory of the spaces $B^{s,s'}_{pq}(U)^{loc}$ a 2-microlocal domain for a given function $f \in \mathcal{S}'(\mathbb{R}^n)$ as in [Mey97].

Definition 7.9: *Let $U \subset \mathbb{R}^n$ compact and $f \in \mathcal{S}'(\mathbb{R}^n)$, then for fixed $0 < p, q \le \infty$*

$$E_{pq}(f,U) = \{(s,s') \in \mathbb{R}^2 : f \in B^{s,s'}_{pq}(U)^{loc}\}$$

defines the 2-microlocal domain.

We have generalized the 2-microlocal domain from [Mey97] and [LVSeu04] which studied the case $p = q = \infty$. We get from the embedding Theorem 7.7 the following.

Lemma 7.10: *Let $(s,s') \in E_{pq}(f,U)$ and let*

$$t \le s \quad \text{and} \quad t + t' \le s + s' \ ,$$

then $(t,t') \in E_{pq}(f,U)$.

Moreover, we are able to proof that this domain is convex.

Lemma 7.11: *The 2-microlocal domain is convex. This means if $(s,s') \in E_{pq}(f,U)$ and $(t,t') \in E_{pq}(f,U)$ then $(\lambda s + (1-\lambda)t, \lambda s' + (1-\lambda)t') \in E_{pq}(f,U)$ for all $\lambda \in [0,1]$.*

Proof: Suppose $f \in B^{s,s'}_{pq}(U)^{loc} \cap B^{t,t'}_{pq}(\mathbb{R}^n,U)^{loc}$ then we can find for both spaces a $V_0 \supset U$ such that we find two functions $g_s \in B^{s,s'}_{pq}(U)^{loc}$ and $g_t \in B^{t,t'}_{pq}(\mathbb{R}^n,U)^{loc}$ with $f = g_s$ on V_0 and $f = g_t$ on V_0. Since V_0 is universal for both spaces, we get from Theorem 7.4 an $l \in \mathbb{N}_0$ and an $A > 0$ and

$$\left(\sum_{\nu=l}^{\infty} 2^{\nu(s-n/p)q} \sum_{G \in G^{\nu,l}} \left(\sum_{m \in U_\nu} |\lambda^{\nu,l}_{Gm}(f)|^p (1 + 2^\nu \operatorname{dist}(2^{-\nu}m, U))^{s'p} \right)^{q/p} \right)^{1/q} < \infty \ ,$$

and

$$\left(\sum_{\nu=l}^{\infty} 2^{\nu(t-n/p)q} \sum_{G \in G^{\nu,l}} \left(\sum_{m \in U_\nu} |\lambda^{\nu,l}_{Gm}(f)|^p (1 + 2^\nu \operatorname{dist}(2^{-\nu}m, U))^{t'p} \right)^{q/p} \right)^{1/q} < \infty$$

where

$$U_\nu = \{m \in \mathbb{Z}^n : \operatorname{dist}(2^{-\nu}m, U) \le A\} \ .$$

Now, in using two times Hölder's inequality we get for arbitrary $\lambda \in [0,1]$

$$\begin{aligned} &\sum_{\nu=l}^{\infty} 2^{\nu(\lambda s + (1-\lambda)t - n/p)q} \sum_{G \in G^{\nu,l}} \left(\sum_{m \in U_\nu} |\lambda^{\nu,l}_{Gm}(f)|^p (1 + 2^\nu \operatorname{dist}(2^{-\nu}m, U))^{(\lambda s' + (1-\lambda)t')p} \right)^{q/p} \\ &= \sum_{\nu=l}^{\infty} 2^{\nu(\lambda s + (1-\lambda)t - n/p)q} \sum_{G \in G^{\nu,l}} \Bigg(\sum_{m \in U_\nu} |\lambda^{\nu,l}_{Gm}(f)|^{\lambda p} (1 + 2^\nu \operatorname{dist}(2^{-\nu}m, U))^{\lambda s' p} \times \\ &\qquad \times |\lambda^{\nu,l}_{Gm}(f)|^{(1-\lambda)p} (1 + 2^\nu \operatorname{dist}(2^{-\nu}m, U))^{(1-\lambda)t'p} \Bigg)^{q/p} \end{aligned}$$

$$\leq \sum_{\nu=l}^{\infty} 2^{\nu(s-n/p)\lambda q} \sum_{G\in G^{\nu,l}} \left(\sum_{m\in U_\nu} |\lambda^{\nu,l}_{Gm}(f)|^p (1+2^\nu \operatorname{dist}(2^{-\nu}m,U))^{s'p} \right)^{\lambda q/p} \times$$

$$\times\, 2^{\nu(t-n/p)(1-\lambda)q} \left(\sum_{m\in U_\nu} |\lambda^{\nu,l}_{Gm}(f)|^p (1+2^\nu \operatorname{dist}(2^{-\nu}m,U))^{t'p} \right)^{(1-\lambda)q/p}$$

$$\leq \left(\sum_{\nu=l}^{\infty} 2^{\nu(s-n/p)q} \sum_{G\in G^{\nu,l}} \left(\sum_{m\in U_\nu} |\lambda^{\nu,l}_{Gm}(f)|^p (1+2^\nu \operatorname{dist}(2^{-\nu}m,U))^{s'p} \right)^{q/p} \right)^{\lambda} \times$$

$$\times \left(\sum_{\nu=l}^{\infty} 2^{\nu(t-n/p)q} \sum_{G\in G^{\nu,l}} \left(\sum_{m\in U_\nu} |\lambda^{\nu,l}_{Gm}(f)|^p (1+2^\nu \operatorname{dist}(2^{-\nu}m,U))^{t'p} \right)^{q/p} \right)^{1-\lambda} < \infty \ .$$

That gives us $f \in B^{\lambda s+(1-\lambda)t,\lambda s'+(1-\lambda)t'}_{pq}(U)^{loc}$. $\square$

Remark 7.12: This 2-microlocal domain clearly gives us new information. For example take the delta distribution and $U \subset \mathbb{R}^n$ compact with $0 \in U$. Then we have for $0 < q < \infty$

$$\delta \in B^{s,s'}_{pq}(U)^{loc} \Leftrightarrow s < \frac{n}{p} - n$$

and for $q = \infty$

$$\delta \in B^{s,s'}_{p\infty}(U)^{loc} \Leftrightarrow s \leq \frac{n}{p} - n \ .$$

Hence, one easily recognices the role played by the values of p and, less important, q (see also Theorem 2.30).

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Lebenslauf

Name	Henning Kempka
Geburtsdatum	22. Mai 1979
Geburtsort	Jena
Staatsbürgerschaft	deutsch
Familienstand	verheiratet seit dem 08.07.2006
Abitur	1998 im Körnberggymnasium Friedrichroda
Armee	1998 bis 1999 im Fallschirmjägerbatallion 263 in Zweibrücken
Studium	1999 bis 2004 Mathematik Diplom mit Nebenfach Informatik an der Friedrich Schiller Universität Jena
Diplom	August 2004
Promotion	seit 2004 Mitarbeiter (befristet) am *mathematischen Institut* Arbeitsgruppe *Funktionenräume*

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